# Financial Flexibility under Financing Constraints and Non-Exclusive Lending

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**Abstract**

This paper examines optimal financial flexibility for firms under financing constraints and non-exclusive lending. We develop a model in which a borrower requires funding for an initial investment and seeks additional financing later following a privately observed liquidity shock. Non-exclusivity creates incentives to dilute initial debt, leading to excessive total borrowing. The optimal contract is an endogenous debt limit: firms with mild shocks retain flexibility but over-borrow (relative to second-best), while those with severe shocks face binding constraints and under-borrow. This limit optimally trades off debt dilution against liquidity needs.

**Keywords:** financial flexibility, debt dilution, lending non-exclusivity, covenant, optimal delegation

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*High yield covenants always seek to strike a delicate balance ... On the one hand, the covenants provide protection for high yield investors against an issuer's overextending itself or unwisely using cash ... On the other hand, the covenants must provide flexibility for the issuer to operate its business and grow ... In other words, the covenants protect the investors' ability to be paid principal and interest ... while preserving the issuer's ability to run its business and grow without undue restrictions.* (Simpson Thacher & Bartlett LLP, 2022)

# 1 Introduction

Financial flexibility refers to a firm's ability to raise financing when needed. It plays a central role in corporate finance because it provides discretion over the timing and amount of borrowing. This discretion is particularly valuable when firms face difficulties in credibly communicating their financing needs to external financiers, such as creditors. In such situations, pre-arranged financial flexibility acts as a safeguard, enabling firms to maintain access to funding.

While valuable, excessive financial flexibility can be harmful to the firm value. For instance, firms with free cash flows may undertake negative-NPV investments (Jensen, 1986). A recent literature in dynamic capital structure argues that excessive flexibility has another cost: it induces firms to persistently increase leverage despite being suboptimal, thereby eroding the value from borrowing (Admati et al., 2018; DeMarzo and He, 2016). This problem occurs because the borrower fails to internalize how additional borrowing dilutes existing creditors' claims.

This paper studies optimal financial flexibility for firms that borrow from multiple lenders (non-exclusive lending). Financial flexibility is desirable because it allows firms to meet future liquidity needs. However, firms face a financing constraint: they must take on initial debt to fund upfront investments. This creates incentives to subsequently dilute the initial debt under financial flexibility, leading to over-borrowing. We show that the optimal contract imposes an endogenous borrowing limit that balances future liquidity needs against dilution incentives, even though this sometimes results in under-borrowing.

More specifically, we develop a model in which, at the initial date, a penniless borrower needs to borrow a fixed amount to finance an investment project. The project generates cash flows at the final date contingent on the borrower's effort, with more debt reducing the borrower's incentive to exert effort. At the interim date, the borrower experiences a liquidity shock, which incentivizes her to borrow further. The second-best solution serves as our benchmark, where liquidity shocks are contractible, and the only friction is moral hazard in effort choice. We focus on the case where the liquidity shock realization is the borrower's private information and she can borrow again afterward.

Crucially, this subsequent borrowing is non-exclusive: the borrower is not obligated to return to initial lenders. This non-exclusivity creates over-borrowing incentives and calls for restrictions on financial flexibility.

The need for upfront borrowing requires the borrower to take on initial debt, while the liquidity shock creates incentives for additional borrowing at the interim date. Initial lenders, anticipating potential dilution from additional borrowing, naturally benefit from protections that restrict total debt. However, these restrictions can be costly when the liquidity shock is severe. We show that our problem resembles a delegation problem between a principal and an agent, where the mechanism does not allow for transfers. In this context, the principal is the borrower at the initial date, and the agent is the same borrower at the interim date after experiencing a liquidity shock. While the principal maximizes total firm value, the agent (with private information about the shock) maximizes the continuation value at the interim date, which is the firm value net of initial debt. Unlike standard delegation problems where conflict is derived from present bias, in our setting, it occurs due to the pre-existing initial debt. As Admati et al. (2018) show, this legacy debt creates a "leverage ratchet" effect where firms persistently increase leverage despite this being suboptimal.

Building on the delegation literature, we show that the optimal contract between the borrower and initial lenders includes an endogenous total debt limit. The borrower may raise additional financing at the interim date, but total debt cannot exceed a pre-specified cap. This debt limit represents what we term financial flexibility. When the liquidity shock is mild, borrowing needs are modest and the debt limit turns out slack. The firm borrows as if under full flexibility. When the shock is severe, however, the borrower would prefer to raise more debt than the cap permits. The limit binds, which restricts the firm's access to funds. The endogenous level of the debt limit is set at the initial date to balance the value of interim liquidity provision against the need to prevent excessive borrowing and creditor dilution.

Why does the optimal contract include an endogenous debt limit? Since the liquidity shock is the borrower's private information, the optimal contract must provide incentives for truthful reporting. This requires that whenever the debt limit does not bind, the borrower can borrow exactly the amount she would choose under full flexibility. When the debt limit binds, however, the borrower cannot benefit from misreporting her liquidity shock. Under this policy, deviations in reported liquidity shock do not increase borrowing capacity because the same debt limit applies to all borrower types. This mechanism–where the borrower either receives her unconstrained optimum or faces a locally flat allocation–follows standard results from optimal delegation theory.

We show that the financing constraint always binds: the borrower raises exactly the amount needed for the initial investment and no more. This occurs because more initial borrowing comes with more debt taken at the initial date. However, due to the dilution incentives, more initial debt must be accompanied by reduced financial flexibility, i.e., a tighter debt limit, which is costly. Therefore, the borrower never raises more than necessary at the initial date. By contrast, the financing constraint might be slack under exclusive lending, where firms must return to the initial lender for additional financing. This occurs when the initial investment is relatively small. In such cases, private information about liquidity shocks introduces no distortion relative to the second-best allocation.

Both over- and under-borrowing exist under the optimal contract. Borrowers who face severe liquidity shocks are constrained by the debt limit and under-borrow relative to the second-best benchmark. By contrast, borrowers experiencing mild shocks raise more debt than the second-best benchmark, resulting over-borrowing. The endogenous debt limit is set to balance this over-borrowing and under-borrowing.

We show the optimal contract can be implemented using a simple and commonly observed financial arrangement: an initial debt issuance paired with a covenant that limits total leverage. Such covenants are widespread in practice and, as the quotes at the beginning of the introduction illustrate, serve precisely the role highlighted in our analysis. They prevent excessive future borrowing that could dilute existing creditors. Our framework thus provides a theoretical foundation for understanding why these covenants are prevalent.

### Related Literature

Our paper is closely related to a strand of the mechanism design literature on commitment and flexibility (Halac and Yared, 2022, 2014; Amador et al., 2006; Amador and Bagwell, 2013). There are two main differences. First, motivated by the corporate financing settings, we incorporate both an initial financing constraint (requiring the borrower to raise a specific amount at $t = 0$) and an interim wealth constraint. We show these constraints generate important results. Second, while the time inconsistency problem in these papers comes from present bias, in our framework, it occurs because the borrower ignores the impact of her decisions on the value of old debt once it has been issued. This creates a disagreement between the principal's and the agent's objective function. The principal – the borrower before the initial debt has been issued – aims to maximize firm value, whereas the agent – the borrower after the initial debt has been issued – aims to maximize the firm value net of the initial debt. On the technical side, our solution involves a two-dimensional maximization problem: an initial debt level and a debt limit function, and the problem is not jointly convex. We manage to circumvent this by solving the problem sequentially.

Our work also connects to the finance literature on financial flexibility (Gamba and Triantis, 2008), though with different modeling primitives. These models emphasize cash hoarding for investment opportunities, whereas our paper introduces private information and focuses instead on flexibility in future leverage. The paper is also related to the theoretical models on debt covenants (Aghion and Bolton, 1992), but the methodology (mechanism design v.s. incomplete contract) differs fundamentally.

# 2 Model Setup

## 2.1 Borrower, Investment, and Liquidity Shock

There are three dates, $t = 0, 1, 2$, and a risk-neutral borrower. The borrower has a project that requires a fixed investment of $I$ at $t = 0$ but has no resources to finance it, hence the need to borrow. The project generates no cash flow at $t = 1$ and a final cash flow $Y$ at $t = 2$ if successful (zero otherwise). At the beginning of $t = 2$, the borrower privately chooses an effort level $q \in [0, 1]$, where $q$ is also the success probability. The effort incurs a private cost $c(q)$. We assume that $c(0) = 0$ and that $c(q)$ is strictly increasing, strictly convex, and twice continuously differentiable. Moreover, we assume $c'(1) > Y$ to guarantee interior solutions for $q$.

The borrower does not discount cash flows between $t = 0$ and $t = 1$. At $t = 1$, she receives a liquidity shock and discounts the cash flows at $t = 2$ at $\theta$. The realization of $\theta$ is her private information. We assume that $\theta$ is drawn from a set $\Theta = [\underline{\theta}, \bar{\theta}] \subseteq (0, 1]$, with a continuously differentiable density function $g(\theta) > 0$ and cumulative density function $G(\theta)$. Throughout the paper, we make the following standard assumption, which leads to a result that the likelihood ratio $g(\theta)/G(\theta)$ is decreasing.

**Assumption 1.** *The cumulative density function $G(\theta)$ is log-concave.*

## 2.2 Lenders and Contract

There exists a competitive pool of dispersed lenders. Throughout the paper, lenders are risk-neutral and do not discount the future. Under competition, the borrower captures all surplus from lending.

The focus of this paper is to study the optimal lending contract, under which at $t = 0$, lenders make an initial transfer $\tau_0$ to the borrower to cover the initial investment $I$. At $t = 1$, the borrower receives another interim transfer $\tau_1(\hat{\theta})$ at $t = 1$ contingent on the reported liquidity shock $\hat{\theta}$. If the project succeeds at $t = 2$, she will make a state-contingent payment $x_1(\hat{\theta})$. As a matter of terminology, we will interpret the repayment at $t = 2$ as a debt contract and refer to $x_1$ as the face value of debt. For a given shock realization $\theta$, report $\hat{\theta}$, and effort $q$, the borrower's payoff is

$$\tau_0 - I + \tau_1(\hat{\theta}) + \theta \left[ q \left( Y - x_1(\hat{\theta}) \right) - c(q) \right].$$

Since the borrower has no initial wealth and is protected by limited liability, the contract must satisfy the following constraints:

$$\tau_0 \geq I, \qquad \tau_1(\hat{\theta}) \geq 0, \qquad x_1(\hat{\theta}) \in [0, Y]. \tag{1}$$

For the rest of the paper, we sometimes refer to the constraint $\tau_0 \geq I$ as the *financing constraint*.

Note that the borrower seeks financing on two dates: an initial investment $I$ at $t = 0$ and an additional transfer $\tau_1$ at $t = 1$ due to the liquidity shock. In most parts of the paper, we focus on the case where *initial* lenders at $t = 0$ are separate from the *new* lenders at $t = 1$. This separation occurs naturally when lenders are dispersed because finding the same group of initial lenders at $t = 1$ can be difficult. Even if the initial lenders are concentrated (e.g., a bank), regulatory capital constraints may restrict their exposure to a particular sector or borrower, preventing them from providing additional funding at $t = 1$. We explicitly distinguish between debt issued to initial lenders ($x_0$) and those issued to new lenders ($x_1 - x_0$).

Figure 1 summarizes the sequence of events on three dates.

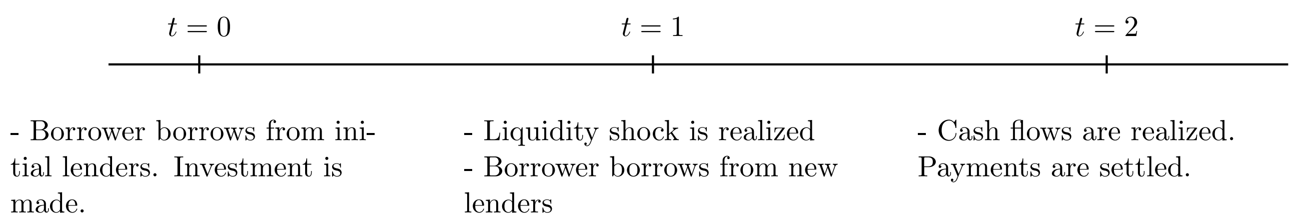


Figure 1: Timeline of events

## 2.3 Effort Choice and Valuations

The borrower's choice of effort at $t = 2$ involves a standard moral hazard problem, which is introduced so that leverage has a downside.[1] Given a face value of debt $x_1$, the borrower chooses effort $q$ to maximize her payoff

$$V_2(x_1) = \max_{q \in [0,1]} q(Y - x_1) - c(q), \tag{2}$$

where $V_2(x_1)$ is the value of the borrower's claim net the effort cost. Because there is no outsider equity, $\theta V_2(x_1)$ can be thought of as the firm's insider equity value at $t = 1$. Let $q(x_1)$ be the optimal

[1] The solution without a moral hazard problem is trivial: the borrower always pledges all the cash flows at $t = 0$. An alternative assumption is to introduce the bankruptcy cost.

solution to this maximization problem. The following lemma summarizes some useful properties satisfied by $V_2(x_1)$ and $q(x_1)$.

**Lemma 1.** *Suppose that, for each $x_1$, $q(x_1)$ solves (2). Then*

- *$\forall x_1 \in [0, Y]$, $V_2(x_1)$ is decreasing and convex with derivative $V_2'(x_1) = -q(x_1)$.*
- *$\forall x_1 \in [0, Y]$, $q(x_1) = (c')^{-1}(Y - x_1)$.*
- *$\forall x_1 \in [0, Y]$, $q(x_1)$ is decreasing with derivative $q'(x_1) = -\frac{1}{c''(q(x_1))}$.*
- *If $c(q)$ is three times continuously differentiable, then $q''(x_1) = -\frac{c'''(q(x_1))}{c''(q(x_1))}\big(q'(x_1)\big)^2$.*

Throughout the paper, we make the following assumptions.

**Assumption 2.** *The cost function $c(q)$ satisfies*

$$I^{\max} \equiv \max_{x_1 \in [0,Y]} \; x_1 q(x_1) \geq I. \tag{3}$$

**Assumption 3.** *$c(q)$ is three times continuously differentiable and satisfies $c'''(q) \geq 0$.*

Assumption 2 states that the maximum expected repayment $x_1 q(x_1)$ is sufficient to cover the initial investment $I$. Assumption 3 and Lemma 1 imply that $q(x_1)$ is concave. This will guarantee that the objective function of several of the programs we later consider is concave.

For the rest of this paper, we define

$$W_1(x_1, \theta) \equiv x_1 q(x_1) + \theta V_2(x_1), \tag{4}$$

as the (ex-post) $t = 1$ firm value for a given $\theta$. Let $x_1^*(\theta)$ be the level of debt that maximizes the firm value, i.e.,

$$x_1^*(\theta) = \arg\max_{x_1} \; W_1(x_1, \theta). \tag{5}$$

This solution corresponds to the *second-best allocation*: it maximizes total surplus when moral hazard is the only friction, and no other constraints are present. This level of debt captures the trade-off between providing liquidity and preserving the borrower's incentives to exert effort. It will be used as a baseline for comparison throughout the remainder of the paper. Because $\frac{\partial^2 W_1(x_1,\theta)}{\partial\theta\partial x_1} < 0$, it is easily established that $x_1^*(\theta)$ decreases with $\theta$: under the second-best allocation, firms with larger liquidity shocks should borrow a higher level of total debt.

This second-best allocation corresponds to the solution after knowing $\theta$. If the borrower chooses at $t = 0$ without knowing $\theta$, the optimal choice is $x_1^*(\mathbb{E}[\theta])$, because $W_1(x_1, \theta)$ is linear in $\theta$. For the remainder of this paper, we sometimes refer to $x_1^*(\mathbb{E}[\theta])$ as the optimal total debt without knowing $\theta$.

**Linear Example.** *Throughout our analysis, we study a specific example where the cost function $c(q)$ is quadratic. Under this specification, the optimal effort $q(x_1)$ is linear in $x_1$, and the borrower's payoff at $t = 2$ is quadratic. For the rest of this paper, we will refer to this case as the linear example. Specifically, if $c(q) = \frac{1}{2}cq^2$ where $c > Y$, then*

$$\begin{aligned} q(x_1) &= \frac{Y - x_1}{c} \\ V_2(x_1) &= \frac{1}{2}\frac{(Y - x_1)^2}{c}. \end{aligned} \tag{6}$$

*Assumption 2 is satisfied only if*

$$\frac{Y^2}{4c} \geq I.$$

*Moreover, we have*

$$\begin{aligned} W_1(x_1, \theta) &= x_1 \frac{Y - x_1}{c} + \frac{\theta}{2}\frac{(Y - x_1)^2}{c} \\ x_1^*(\theta) &= \frac{1 - \theta}{2 - \theta} Y. \end{aligned}$$

## 2.4 Complete Information Solution

If $\theta$ is observable and contractible, the optimal long-term contract is designed to maximize the borrower's expected payoff, subject to the lender's participation constraint, financing, and feasibility constraints.

$$\begin{aligned} &\max_{\{\tau_0, \tau_1(\theta), x_1(\theta)\}} \; \tau_0 - I + \mathbb{E}\Big[\tau_1(\theta) + \theta V_2\left(x_1(\theta)\right)\Big] \\ &\text{subject to} \\ &\mathbb{E}\Big[x_1(\theta) q\left(x_1(\theta)\right) - \tau_1(\theta)\Big] - \tau_0 \geq 0 \\ &\tau_0 \geq I, \qquad \tau_1(\theta) \geq 0, \qquad x_1(\theta) \in [0, Y]. \end{aligned} \tag{7}$$

Clearly, the lender's participation constraint is always binding, and it is without loss of generality to set $\tau_1(\theta) = 0$. We substitute the participation constraint in the objective function and formulate Lagrangian. Let $\lambda$ be the Lagrange multiplier of the financing constraint. Dividing the Lagrangian by $1+\lambda$, the problem becomes

$$\min_{\lambda \geq 0} \max_{\{x_1(\theta)\}} \mathbb{E}\Big[W_1\left(x_1(\theta), \theta\right) - \lambda \frac{\theta}{1+\lambda} V_2\left(x_1(\theta)\right)\Big] - I, \tag{8}$$

**Lemma 2** (Complete Information Benchmark)**.** *Let*

$$I^c \equiv \mathbb{E}\Big[x_1^*(\theta) q\left(x_1^*(\theta)\right)\Big].$$

*When $\theta$ is observable and contractible, it is without loss of generality to set $\tau_1(\theta) = 0$.*

- *If $I^c \geq I$, the financing constraint is slack, and the optimal solution satisfies $x_1^{c*}(\theta) = x_1^*(\theta)$ and $\tau_0 = I^c$.*
- *Otherwise, the solution is $x_1^{c*}(\theta) = x_1^*\left(\delta^{c*}\theta\right)$ where $\delta^{c*} \equiv 1/(1+\lambda^{c*})$ solves*

$$\mathbb{E}\left[x_1^*\left(\delta^{c*}\theta\right) q\left(x_1^*\left(\delta^{c*}\theta\right)\right)\right] = I.$$

*In this case, $\tau_0 = I$.*

The term $\mathbb{E}[x_1^*(\theta) q(x_1^*(\theta))]$ measures the expected pledgeable income at $t=0$ under the second-best allocation. Lemma 2 presents two distinct cases based on whether this expected pledgeable income covers the initial investment $I$. When this expected pledgeable income is sufficient, the financing constraint is slack, and the second-best allocation is optimal. When insufficient, the financing constraint binds, so the borrower must pledge more project income. Equation (8) shows this scenario is equivalent to the second-best allocation where the borrower faces an effective liquidity shock $\delta^{c*}\theta$. Results show that the financing constraint, when binding, amplifies the borrower's liquidity shock so that a borrower with shock $\theta$ effectively behaves as if facing a larger shock $\delta^{c*}\theta$. As a result, it leads to excessive borrowing and total debt relative to the second-best solution.

Before concluding this subsection, we note that the results are equivalent under non-exclusive lending. Depending on whether the financing constraint binds, the optimal contract specifies total debt due at $t=2$ as $x_1^*(\theta)$ or $x_1^*(\delta^{c*}\theta)$. Intuitively, if total debt can be contracted without incurring incentive rents, there is no associated dilution.

**Linear Example.** *It is easily derived that*

$$\mathbb{E}\left[x_1^*\left(\delta\theta\right) q\left(x_1^*\left(\delta\theta\right)\right)\right] = \frac{Y^2}{c} \mathbb{E}\left[\frac{1-\delta\theta}{\left(2-\delta\theta\right)^2}\right].$$

*If we further consider the special case of the uniform distribution:* $G(\theta) = \frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}}$*, then*

$$\frac{Y^2}{c}\mathbb{E}\left[\frac{1-\delta\theta}{(2-\delta\theta)^2}\right] = \frac{1}{\bar{\theta}-\underline{\theta}}\frac{Y^2}{\delta c}\left[\frac{1}{2-\delta\underline{\theta}} - \frac{1}{2-\delta\bar{\theta}} - \log\left(\frac{2-\delta\bar{\theta}}{2-\delta\underline{\theta}}\right)\right].$$

*The financing constraint is slack if*

$$\frac{1}{\bar{\theta}-\underline{\theta}}\frac{Y^2}{c}\left[\frac{1}{2-\underline{\theta}} - \frac{1}{2-\bar{\theta}} - \log\left(\frac{2-\bar{\theta}}{2-\underline{\theta}}\right)\right] \geq I.$$

# 3 Benchmark: Single Lender Equilibrium

Before solving the model, we first analyze a benchmark case where the borrower can enter into a long-term contract with a single lender. This corresponds to the case of exclusive lending. In this case, there are no new lenders, and initial lenders receive a payoff

$$x_1(\theta)q\left(x_1(\theta)\right) - \tau_1(\theta) - \tau_0.$$

In addition to moral hazard, the remaining frictions are the financing constraint and the borrower's private information about her liquidity shock. We will analyze the impact of each of these frictions separately.

## 3.1 Private Information

Next, we turn to the case where $\theta$ is private information. By the revelation principle, we can restrict attention to direct revelation mechanisms $\{\tau_0, \tau_1(\theta), x_1(\theta)\}$. The optimal mechanism can be found by solving the problem in (7) with the additional truth-telling IC constraint

$$\theta \in \arg\max_{\hat{\theta}} \; \tau_1(\hat{\theta}) + \theta V_2(x_1(\hat{\theta})). \tag{9}$$

By the envelope theorem, we can rewrite (9) as

$$\begin{aligned} &\tau_1(\theta) + \theta V_2\left(x_1(\theta)\right) = \tau_1(\bar{\theta}) + \bar{\theta}V_2(x_1(\bar{\theta})) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta}, \; \forall \theta \in \Theta \\ &x_1(\theta) \text{ is non-increasing.} \end{aligned} \tag{10}$$

As in the problem under complete information, the lender's $t = 0$ participation constraint must be binding. Simple calculation shows that the initial financing constraint becomes [2]

$$\tau_0 = \mathbb{E}\left[W_1(x_1(\theta), \theta) + \frac{G(\theta)}{g(\theta)} V_2\left(x_1(\theta)\right)\right] - \tau_1(\bar{\theta}) - \bar{\theta} V_2(x_1(\bar{\theta})) \geq I. \tag{11}$$

Substituting $\tau_0$ and $\tau_1(\theta)$ into the objective function, the optimal contract solves[3]

$$\begin{aligned} &\max_{\{x_1(\theta), \tau_1(\theta) \geq 0\}} \ \mathbb{E}\left[W_1(x_1(\theta), \theta)\right] - I \\ &\text{subject to (10) and (11)} \end{aligned} \tag{12}$$

We make the following assumption to ensure that the previous optimization problem is convex:

**Assumption 4.** *For all $\theta \in \Theta$, the function*

$$W_1(x_1, \theta) + \frac{G(\theta)}{g(\theta)} V_2(x_1)$$

*is concave in $x_1$. Under Assumption 1, this holds if*

$$2 + \frac{x_1 q''(x_1)}{q'(x_1)} \geq \bar{\theta} + \frac{1}{g(\bar{\theta})}.$$

Note that problem (12) differs from the standard mechanism design problem in that it has an additional constraint $\tau_1(\theta) > 0$. Lemma 3 in the appendix proves that $\tau_1(\theta)$ must be non-increasing, which implies that we only need to evaluate the constraint at $\theta = \bar{\theta}$. Due to the financing constraint, it is costly to defer the borrowing from $t = 0$ to $t = 1$, so that the principal, in general, prefers lower $\tau_1(\theta)$. This also implies $\tau_1(\bar{\theta}) = 0$ if the financing constraint is binding.[4]

We solve the problem in two steps. First, with some slight abuse of notation, let $\lambda$ be the Lagrange multiplier of the financing constraint $\tau_0 \geq I$ and $\delta \equiv 1/(1+\lambda)$ be the shadow discount

[2] This involves the following simplification by changing the order of integration:

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta})) g(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\tilde{\theta}} V_2(x_1(\tilde{\theta})) g(\theta) d\theta d\tilde{\theta} = \int_{\underline{\theta}}^{\bar{\theta}} V_2(x_1(\tilde{\theta})) G(\tilde{\theta}) d\tilde{\theta}.$$

The problem also substitutes the envelope condition (10) in the participation constraint.

[3] The constraint $x_1(\theta) \in [0, Y]$ will always be slack and hence omitted.

[4] If the financing constraint is slack, the borrower is indifferent between borrowing at $t = 0$ and $t = 1$ due to no discount between the two dates. As long as there is a slight preference for $t = 0$ borrowing, $\tau_1(\bar{\theta}) = 0$ would hold.

factor. We find the optimal solution for a fixed value of $\delta$:

$$\begin{aligned}
&\max_{\{x_1(\theta)\}} \ \mathbb{E}\left[W_1(x_1(\theta),\theta)+(1-\delta)\frac{G(\theta)}{g(\theta)}V_2(x_1(\theta))\right]-(1-\delta)\bar{\theta}V_2(x_1(\bar{\theta}))\\
&\text{subject to}\\
&x_1(\theta) \text{ is non-increasing}
\end{aligned} \tag{13}$$

Note that the objective function in equation (13) differs from standard mechanism design problems because it includes a final term involving $\bar{\theta}$. Therefore, we cannot apply the standard point-wise maximization technique. To see this, let

$$x_1^s(\theta) \equiv \arg\max_{x_1}\left\{W_1(x_1,\theta)+(1-\delta)\frac{G(\theta)}{g(\theta)}V_2(x_1)\right\} \tag{14}$$

be the point-wise maximum of (13). Note that $x_1^s(\theta) = x_1^*(\theta)$ when $\delta = 1$. Moreover, under Assumption 1, the objective function in (14) satisfies the single-crossing property, so that $x_1^s(\theta)$ is non-increasing in $\theta$. However, $x_1(\bar{\theta})$ is chosen to maximize

$$W_1(x_1(\bar{\theta}),\bar{\theta})+(1-\delta)\frac{G(\bar{\theta})}{g(\bar{\theta})}V_2(x_1(\bar{\theta}))-(1-\delta)\bar{\theta}V_2(x_1(\bar{\theta})),$$

so that the point-wise maximum solution will lead to a solution where $x_1(\theta)$ has an upward jump at $\bar{\theta}$. Using the method from global theory of constrained optimization, we have the following result.

**Proposition 1.** *Let $x_1^{s*}(\theta)$ be the solution to problem (13) for a given $\delta$.*

- *If $\delta \geq \underline{\theta}/\mathbb{E}[\theta]$, the solution to problem (13) is $x_1^{s*}(\theta) = \max\{x_1^s(\theta), x_1^s(\theta_H)\}$ where $\theta_H \in \Theta$ solves:*

$$\int_{\theta_H}^{\bar{\theta}} \frac{\partial W_1(x_1^s(\theta_H),\theta)}{\partial x_1} dG(\theta) + (1-\delta)q\left(x_1^s(\theta_H)\right)\left(\theta_H G(\theta_H)+\int_{\theta_H}^{\bar{\theta}}\theta dG(\theta)\right)=0. \tag{15}$$

- *If $\delta < \underline{\theta}/\mathbb{E}[\theta]$, $x_1^{s*}(\theta) = x_1^p$, where $x_1^p$ solves*

$$x_1^p q'(x_1^p) + (1-\delta\mathbb{E}[\theta])\, q(x_1^p) = 0.$$

*The optimal transfers $\{\tau_0, \tau_1(\theta)\}$ satisfy (10) and (11).*

Proposition 1 describes the optimal level of debt under different degrees of financing constraint. When $\delta \geq \underline{\theta}/\mathbb{E}[\theta]$, the financing constraint is not too tight or may even be slack. In this case, the optimal debt level decreases with $\theta$ for lower values of $\theta$ (following $x_1^s(\theta)$) and then remains constant at $x_1^s(\theta_H)$ once $\theta$ exceeds the threshold $\theta_H$. In contrast, when $\delta < \underline{\theta}/\mathbb{E}[\theta]$, the financing constraint becomes very tight. In this case, the optimal debt level is a constant at $x_1^p$ across all types $\theta$. In both cases, the optimal debt could involve a flat region once $\theta$ gets sufficiently high. Let us elaborate.

Borrowers with higher values of $\theta$ inherently need less debt. The incentive compatibility constraint then requires that the interim payment $\tau_1$ must also decrease for borrowers with higher $\theta$. This creates a natural downward-sloping pattern in both the debt level and interim transfers across types. Given the financing constraint, the principal – the borrower at $t = 0$ – prefers to pledge more of the future cash flows, which necessarily involves a lower $\tau_1$. Because $\tau_1$ decreases with $\theta$, this eventually causes higher types to hit the limited liability constraint $\tau_1(\theta) \geq 0$. Once this constraint binds for types above $\theta_H$, these borrowers cannot make any interim payment ($\tau_1 = 0$). To maintain incentive compatibility when the limited liability constraint binds, the face value of debt $x_1$ must also remain constant for all types above $\theta_H$.

We now proceed to the second step on the optimal solution.

**Proposition 2.** *Let*

$$I^s \equiv \mathbb{E}\left[W_1\left(x_1^*(\theta), \theta\right) + V_2\left(x_1^*(\theta)\right)\frac{G(\theta)}{g(\theta)}\right] - \bar{\theta}V_2\left(x_1^*(\bar{\theta})\right),$$

*where $x_1^*(\theta)$ is defined in (5). Then,*

- *If $I \leq I^s$, the optimal policy is $x_1^*(\theta)$.*
- *If $I > I^s$, the optimal policy is $x_1^{s*}(\theta)$, where $\{\theta_H, \delta^{s*}\}$ jointly solve equations (15) and $\tau_0 = I$.*

This proposition characterizes how the optimal solution depends on the investment size and, hence, the financing needs. When the financing constraint is slack (which corresponds to $I \leq I^s$), the optimal solution is identical to the second-best allocation, i.e., the borrower's private information about $\theta$ does not introduce any additional distortion in the debt choice. This is because, in the absence of financing constraints, there is no conflict between the principal and the agent: both aim to maximize the firm value.

However, when the financing constraint binds (i.e., when $I > I^s$), the truth-telling incentive compatibility constraint introduces additional distortions. As shown by (14), the extent of this distortion depends on $\delta$, which captures the severity of the financing constraint. The solution then is directly linked to the two cases in Proposition 1, where the face value of debt involves a flat region.

**Linear Example.** *It is easily derived that*

$$x_1^s(\theta) = Y\left[\frac{1-\theta-(1-\delta)\frac{G(\theta)}{g(\theta)}}{2-\theta-(1-\delta)\frac{G(\theta)}{g(\theta)}}\right].$$

*If $G(\theta)$ is log-concave, we have that Assumption 4 holds if*

$$(2-\bar{\theta})g(\bar{\theta}) \geq 1.$$

*Substituting in the envelope condition, we get that for $\theta \leq \theta_H$ the transfer to the borrower at $t = 1$ is*

$$\tau_1(\theta) = \frac{Y^2}{2c}\left[\frac{\theta_H}{\left(2-\theta_H-(1-\delta)\frac{G(\theta_H)}{g(\theta_H)}\right)^2} - \frac{\theta}{\left(2-\theta-(1-\delta)\frac{G(\theta)}{g(\theta)}\right)^2} - \int_\theta^{\theta_H}\frac{1}{\left(2-\tilde{\theta}-(1-\delta)\frac{G(\tilde{\theta})}{g(\tilde{\theta})}\right)^2}d\tilde{\theta}\right].$$

*Moreover, we get*

$$x_1^P = \frac{1-\delta\mathbb{E}[\theta]}{2-\delta\mathbb{E}[\theta]}.$$

*If we further assume that $\theta \sim U[\underline{\theta}, \bar{\theta}]$, we get that*

$$\begin{aligned}
&x_1^s(\theta) = Y\left[1-\frac{1}{\alpha-\beta\theta}\right] \\
&\tau_1(\theta) = \frac{Y^2}{2c}\left[\frac{\theta_H}{(\alpha-\beta\theta_H)^2} - \frac{\theta}{(\alpha-\beta\theta)^2} - \frac{\theta_H-\theta}{(\alpha-\beta\theta_H)(\alpha-\beta\theta)}\right] \\
&\textit{where } \alpha \equiv 2+(1-\delta)\underline{\theta} \textit{ and } \beta = 2-\delta.
\end{aligned}$$

# 4 Multiple Lenders Equilibrium

We now solve the model under non-exclusive lending. In this setting, initial lenders anticipate dilution from new debt issued at $t = 1$ and, therefore, seek to limit future issuance to protect their claims. We consider the contract $\{\tau_0, x_0, x_1(\hat{\theta})\}$ offered to initial lenders, where total debt is capped by $x_1(\hat{\theta})$, a function of the borrower's reported type $\hat{\theta}$ at $t = 1$. By doing so, we restrict the mechanism to those where transfers cannot be made to the initial lenders and adjustments cannot be applied to $x_0$ conditional on the reported type $\hat{\theta}$ at $t = 1$. We examine such mechanisms later in Section 5.

Since the borrower faces a liquidity shock, it is without loss of generality to assume she issues new debt up to the debt limit $x_1(\hat{\theta})$. Thus, $\Delta x_1(\hat{\theta}) \equiv x_1(\hat{\theta}) - x_0$ as the additional debt issued at $t = 1$.

### 4.1 Reduction to Delegation Problem

We first show that the problem can be equivalently formulated as a delegation problem (Amador et al., 2006; Amador and Bagwell, 2013). The participation constraint for new lenders bind, implying $\tau_1 = (x_1 - x_0)q(x_1)$, where $x_1 \geq x_0$ is required to satisfy the non-negativity constraint $\tau_1 \geq 0$. At $t = 1$, the borrower's continuation payoff for any given $(\theta, x_1)$ is:

$$V_1(x_1, x_0, \theta) = \underbrace{(x_1 - x_0)q(x_1)}_{\text{new debt issuance}} + \theta V_2(x_1) = W_1(x_1, \theta) - x_0 q(x_1). \tag{16}$$

The participation constraint for initial lenders also bind, implying

$$\tau_0 = x_0 \mathbb{E}\left[q(x_1(\theta))\right],$$

and the financing constraint becomes

$$x_0 \mathbb{E}\left[q(x_1(\theta))\right] \geq I. \tag{17}$$

Therefore, the borrower's expected payoff at $t = 0$ is given by:

$$\tau_0 - I + \mathbb{E}\left[V_1(x_1(\theta), x_0, \theta)\right] = \mathbb{E}\left[W_1(x_1, \theta)\right] - I.$$

The borrower has incentives to report its type truthfully only if the following incentive compatibility constraint is satisfied

$$\theta \in \arg\max_{\hat{\theta} \in \Theta} V_1(x_1(\hat{\theta}), x_0, \theta).$$

By the envelope theorem, this can be rewritten as

$$\begin{aligned} &V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \quad \forall \theta \in \Theta \\ &x_1(\theta) \text{ is non-increasing.} \end{aligned} \tag{18}$$

The optimal mechanism at $t = 0$ solves the following problem:

$$\begin{gathered} \max_{x_0, x_1(\theta) \geq x_0} \mathbb{E}\Big[W_1(x_1, \theta)\Big] \\ \text{subject to (17) and (18).} \end{gathered} \tag{19}$$

This problem relates to a delegation setup, where the principal is the borrower at time $t = 0$, and the agent is the borrower at time $t = 1$. The principal delegates the choice of total debt to the agent, who has private information about the liquidity shock $\theta$. The principal seeks to maximize firm value (net investment $I$), while the agent maximizes $V_1(x_1, x_0, \theta)$, which, as shown by equation (16), is the firm value net of initial debt at $t = 1$. This conflict relates to the insight of Aguiar et al. (2019), which shows that equilibrium debt issuance decisions can be characterized as the solution to a planner's problem that ignores the payoff to existing creditors.

Our problem differs in several aspects. First, it includes a financing constraint (17), which arises naturally in corporate finance settings. This constraint is later shown to always bind, adding both new insights and analytical complexity to the problem. Second, unlike Amador and Bagwell (2013), where the time-inconsistency problem is driven by present bias, here it exists because the agent disregards the debt issued at $t = 0$, a concern raised by Fama and Miller (1972) and Black and Scholes (1973) and formalized by Admati et al. (2018). Specifically, note that

$$W_1(x_1, \theta) - V_1(x_1, x_0, \theta) = x_0 q(x_1),$$

so the wedge between the principal's and agent's payoffs exactly equals the value of the initial debt.

The optimization problem in (19) is not necessarily jointly convex in $x_0$ and $x_1(\theta)$. Hence, we analyze the problem in two steps. In subsection 4.2, we characterize the optimal debt at $t = 1$ for an exogenously given $x_0$. In subsection 4.3, we solve for the amount of initial debt $x_0$ issued at $t = 0$.

## 4.2 Constrained Debt Limits

Throughout this subsection, we take the initial debt level $x_0$ as given. Let us define the value function

$$\begin{gathered} W_0(x_0) = \max_{x_1(\theta) \geq x_0} \mathbb{E}\Big[W_1(x_1, \theta)\Big] \\ \text{subject to (17) and (18).} \end{gathered} \tag{20}$$

Before proceeding, it is helpful to consider a local version of the incentive compatibility constraint, which applies at any point $\theta$ where $x_1(\theta)$ is differentiable.

$$\frac{\partial V_1\left(x_1(\theta), x_0, \theta\right)}{\partial x_1} \cdot \frac{dx_1(\theta)}{d\theta} = 0. \tag{21}$$

This condition implies that a necessary condition for truthful reporting is either: (i) the borrower receives the unconstrained optimal debt level $x_1(\theta)$ that maximizes her continuation payoff $V_1(x_1(\hat{\theta}), x_0, \theta)$, or (ii) the allowed debt level $x_1(\theta)$ is locally insensitive to the type reported, so the agent does not benefit from locally misreporting the liquidity shock. Regarding the first case, let us define

$$x_1^f(\theta, x_0) = \arg\max_{x_1} \; V_1(x_1, x_0, \theta) \tag{22}$$

as the optimal total debt under *flexible* debt issuance. For notational simplicity, we omit the dependence on $x_0$ and write $x_1^f(\theta)$ when no confusion arises. The first-order condition is:[5]

$$(x_1^f(\theta) - x_0)q'(x_1^f(\theta)) + (1-\theta)q(x_1^f(\theta)) = 0.$$

To simplify the exposition, we make the following assumption:

**Assumption 5.** $\bar{\theta} = 1$.

Under Assumption 5, the most patient borrower does not issues any debt at $t = 1$, so $x_1^f(\bar{\theta}) = x_0$. This assumption eliminates the need to consider limits that constrain all types, reducing the cases to analyze.[6] Moreover, under this assumption, for any debt limit between $x_0$ and $x_1^f(\underline{\theta})$, there exists a threshold type $\theta_L \in \Theta$ such that $x_1^f(\theta_L)$ is exactly this limit. In equilibrium, some borrowers hit the debt limit, allowing us to specify the limit equivalently as a threshold type $\theta_L$.

We now solve (20) and establish conditions under which the optimal mechanism takes the form of a debt limit $x_1^f(\theta_L)$ so that the borrower has total debt $x_1^L(\theta) \equiv \min\{x_1^f(\theta), x_1^f(\theta_L)\}$. By construction, this debt policy satisfies the local incentive compatibility conditions (21) and is also globally incentive compatible (i.e., satisfies (18)). Figure 2 plots the debt policy (solid red line) against the flexible debt level $x_1^f(\theta)$ (dashed red line) and the second-best solution $x_1^*(\theta)$ (solid green line). Several patterns emerge. First, $x_1^f(\theta)$ decreases with $\theta$, so the debt limit binds at low $\theta$ but becomes slack as $\theta$ rises. Second, $x_1^f(\theta) \geq x_1^*(\theta)$, since the former maximizes the firm value

[5] The function $V_1(x_1, x_0, \theta)$ is concave in $x_1$ under Assumption 3.

[6] When $\bar{\theta} < 1$, we would need to consider additional cases to allow for debt limits of the form $0 < x_1^L < x_1^f(\bar{\theta})$. This corresponds to the degenerated delegation sets in Amador et al. (2018). The analysis in the appendix considers this more general case and allows for $\bar{\theta} < 1$.

net of initial debt, while the latter maximizes total firm value. As a result, the debt limit leads to under-borrowing at low $\theta$ and over-borrowing at high $\theta$.

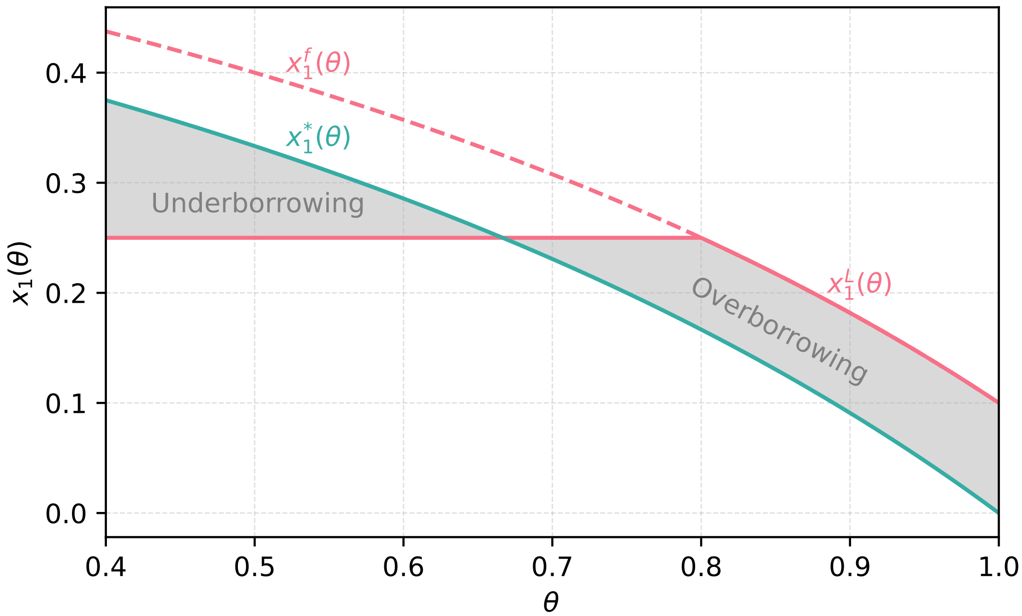


Figure 2: Debt Limit and Second-Best Allocation. We assume that $\theta \sim U[0.4, 1]$, $Y = 1$ and $c(q) = q^2$. For the purpose of illustration, we take $x_0 = 0.1$ and $\theta_L = 0.8$.

The optimal debt limit can thus be found by solving:

$$\begin{aligned}
&\max_{\theta_L \in \Theta} \; \mathbb{E}\left[W_1\left(x_1^f(\theta_L), \theta\right) \mathbf{1}_{\theta \le \theta_L} + W_1\left(x_1^f(\theta), \theta\right) \mathbf{1}_{\theta > \theta_L}\right] \\
&\text{subject to} \\
&x_0 \mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right] \ge I.
\end{aligned} \tag{23}$$

Again, we formulate Lagrangian. Let $\lambda$ be the multiplier and define $\delta = \frac{1}{\lambda+1}$. It is easily established that $\mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right]$ increases in $\theta_L$. This means that we only need to consider two candidates for the solution of problem (23): either (i) the constraint is slack, and $\theta_L$ is chosen to maximize the objective function; or (ii) the constraint is binding and $\theta_L$ is determined by the financing constraint. The following proposition characterizes the solution to problem (23).

**Proposition 3** (Optimal Debt Limit)**.** *Let $\theta_L^{\text{bc}}$ be the lowest type that satisfies the financing constraint:*

$$\theta_L^{\text{bc}} \equiv \min\left\{\theta_L \in \Theta : \; x_0 \mathbb{E}\left[q\big(x_1^f(\theta_L)\big) \mathbf{1}_{\{\theta \le \theta_L\}} + q\big(x_1^f(\theta)\big) \mathbf{1}_{\{\theta > \theta_L\}}\right] \ge I\right\}.$$

*Let $\theta_L^{\mathrm{uc}}$ be the optimal threshold in the absence of the financing constraint.*

- *If $x_0 > x_1^*\big(\mathbb{E}[\theta]\big)$, then $\theta_L^{\mathrm{uc}} = \bar{\theta} = 1$.*
- *If $x_0 \le x_1^*\big(\mathbb{E}[\theta]\big)$, then $\theta_L^{\mathrm{uc}}$ solves*

$$x_0 \cdot \frac{q'\big(x_1^f(\theta_L)\big)}{q\big(x_1^f(\theta_L)\big)} + \mathbb{E}[\theta_L - \theta \mid \theta \le \theta_L] = 0. \tag{24}$$

*The optimal threshold is $\theta_L = \max\left\{\theta_L^{\mathrm{uc}}, \theta_L^{\mathrm{bc}}\right\}$, and the corresponding debt limit is $x_1^L = x_1^f(\theta_L)$.*

- *If $\theta_L^{\mathrm{uc}} > \theta_L^{\mathrm{bc}}$, the financing constraint is slack, and $\delta = 1$.*
- *If $\theta_L^{\mathrm{uc}} \le \theta_L^{\mathrm{bc}}$, the financing constraint binds, and $\delta = -\frac{x_0}{\mathbb{E}[\theta_L^{\mathrm{bc}} - \theta \mid \theta \le \theta_L^{\mathrm{bc}}]} \cdot \frac{q'\left(x_1^f(\theta_L^{\mathrm{bc}})\right)}{q\left(x_1^f(\theta_L^{\mathrm{bc}})\right)}$.*

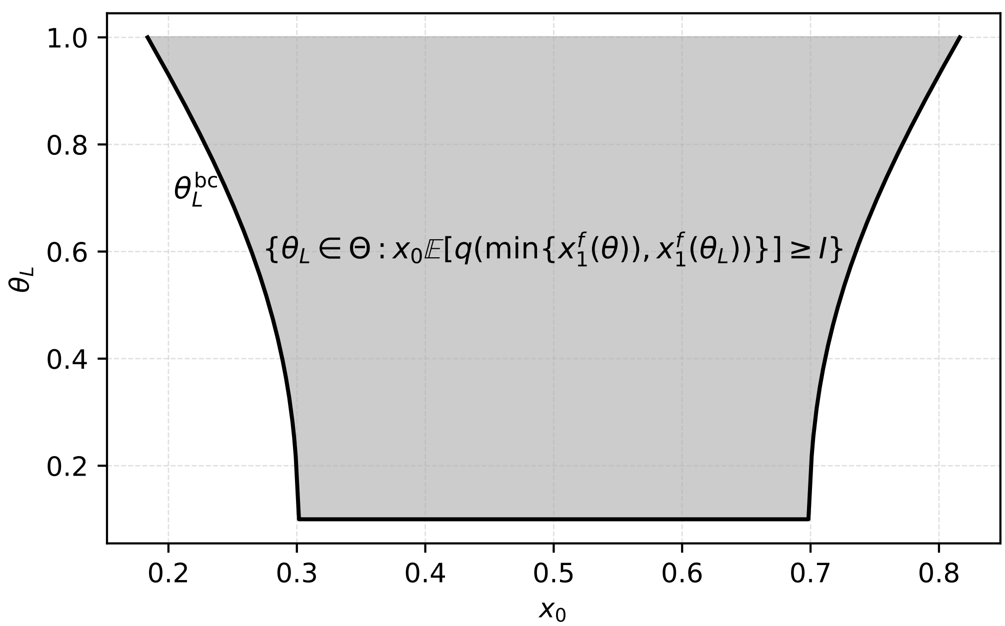


Figure 3: Illustration feasible set for the debt limit problem (23). We assume that $\theta \sim U[0.1, 1]$, $Y = 1$ and $c(q) = q^2$. The maximum pledgeable income is $I = 0.6 \times I^{\max}$ where $I^{\max} = \max_{x_1 \in [0,Y]} x_1 q(x_1)$.

Figure 3 illustrates the set of $(x_0, \theta_L)$ where the borrower can raise at least $I$ at $t = 0$. The shaded region is the feasible set, with lower boundary being $\theta_L^{bc}$. A minimum $x_0$ exists below which raising $I$ is impossible. When $x_0$ is low but above this minimum, increasing $x_0$ decreases

$\theta_L^{bc}$, effectively raising the debt limit at $t = 1$. This happens because to raise $I$, a higher $x_0$ can be accommodated with a lower success probability, which corresponds to a higher level of total debt. As $x_0$ increases further, the constraint becomes slack, because the maximum feasible debt exceeds what even type $\underline{\theta}$ would flexibly take. At very high $x_0$, the pattern reverses: further increases raise $\theta_L^{bc}$, effectively tightening the limit. This occurs because high $x_0$ already creates severe moral hazard in effort choice, reducing initial debt's market value. To preserve this value, a tighter debt is limit is needed so that the expected payment to initial lenders can be at least $I$. In later analysis, the region where $\theta_L^{bc}$ increases with $x_0$ is never optimal in the borrower's problem.

Meanwhile, the threshold $\theta_L^{\text{uc}}$ balances over- and under-borrowing when the financing constraint is ignored. Without knowing $\theta$, the optimal debt level is $x_1^*\big(\mathbb{E}[\theta]\big)$ at $t = 0$. If initial debt $x_0$ exceeds $x_1^*(\mathbb{E}[\theta])$, there is already over-borrowing at $t = 0$, so $\theta_L^{\text{uc}}$ is chosen to impose the tightest possible constraint to reduce over-borrowing. When $x_0$ falls below $x_1^*\big(\mathbb{E}[\theta]\big)$, $\theta_L^{\text{uc}}$ is determined by equation (24), maximizing the objective in (23). As Figure 2 shows, such debt limit causes under-borrowing when $\theta$ is low but over-borrowing when $\theta$ is high. The threshold $\theta_L$ minimizes the combined welfare loss from both distortions.

According to Proposition 3, the optimal policy takes $\theta_L = \max\{\theta_L^{\text{uc}}, \theta_L^{\text{bc}}\}$, acknowledging that the threshold must be at least $\theta_L^{\text{bc}}$ to satisfy the initial financing requirement. When $\theta_L^{\text{uc}} > \theta_L^{\text{bc}}$, the financing constraint does not bind ($\lambda = 0$), whereas when $\theta_L^{\text{uc}} \leq \theta_L^{\text{bc}}$, the constraint binds and generates a positive Lagrange multiplier reflecting the shadow cost of the financing requirement.

The next proposition identifies sufficient conditions under which the debt limit characterized in the proposition solves the mechanism design problem (31). Our proof follows the Lagrangian methodology employed by Amador et al. (2006) and Amador and Bagwell (2013).

**Proposition 4** (Sufficient Conditions for Optimality)**.** *Consider the debt limit in Proposition 3. Let $\delta = 1/(1 + \lambda)$ where $\lambda$ be the Lagrange multiplier of the financing constraint in (23). If the function function*

$$\delta G(\theta) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} g(\theta) \tag{25}$$

*is non-decreasing for $\theta \in [\theta_L, \bar{\theta}]$. Then, $x_1^{m*}(\theta) = \min\{x_1^f(\theta), x_1^f(\theta_L)\}$ solves the mechanism design problem (20).*

Why is a debt limit optimal? The agent ignores how additional borrowing dilutes existing debt $x_0$, leading to over-borrowing. The principal naturally wants to impose discipline. Since the liquidity shock is private information, the agent's incentive compatibility constraint requires that total debt either maximizes the agent's continuation value or is locally flat in the reported type (equation (21)). Condition (25) is a regularity condition that ensures this is also globally optimal.

**Linear Example.** *We have*

$$V_1(x_1, x_0, \theta) = (x_1 - x_0)\frac{Y - x_1}{c} + \frac{\theta}{2}\frac{(Y - x_1)^2}{c}$$
$$x_1^f(\theta, x_0) = \frac{(1-\theta)Y + x_0}{2-\theta}.$$

*The price at $t = 1$ given $x_1^f(\theta, x_0)$ is*

$$q(x_1^f(\theta, x_0)) = \frac{Y - x_0}{c}\frac{1}{2-\theta},$$

*so*

$$x_0\mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right] = \frac{x_0(Y-x_0)}{c}\mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta\leq\theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right].$$

*The sufficient condition (25) in Proposition 4 is satisfied only if*

$$(1-\delta) + \delta\frac{Y}{x_0} > (2-\theta)\frac{g'(\theta)}{g(\theta)} \quad \forall\theta \in [\theta_L, \bar{\theta}].$$

*As $x_0 < Y$, a sufficiency condition is that*

$$\sup_{\theta\in\Theta}(2-\theta)\frac{g'(\theta)}{g(\theta)} \leq 1$$

### 4.3 Initial Debt and Implementation

We now proceed to the second step and find the optimal value of $x_0$ by solving the following problem:

$$\begin{aligned} &\max_{x_0}\ W_0(x_0) \\ &\text{subject to} \\ &x_0 q(x_0) \geq I. \end{aligned} \tag{26}$$

The constraint $x_0 q(x_0) \geq I$ is both sufficient and necessary for the feasible set in problem (20) to be non-empty. It is easily established that this constraint holds if and only if $x_0 \in [x_0^{\min}, x_0^{\max}]$, where $x_0^{\min} < x_0^{\max}$ are the two roots to the equation $x_0 q(x_0) = I$.[7]

Figure 4 illustrates the problem in (26). It builds on the left-hand boundary of the feasible set shown in Figure 3 and adds the borrower's indifference curves to represent her preferences, with darker curves indicating higher payoff. For a given $\theta_L$ (and thus a fixed debt limit), increasing $x_0$

[7] Assumption 2 guarantees the existence of the root. The fact $x_0 q(x_0)$ is concave in $x_0$ implies that there are two roots.

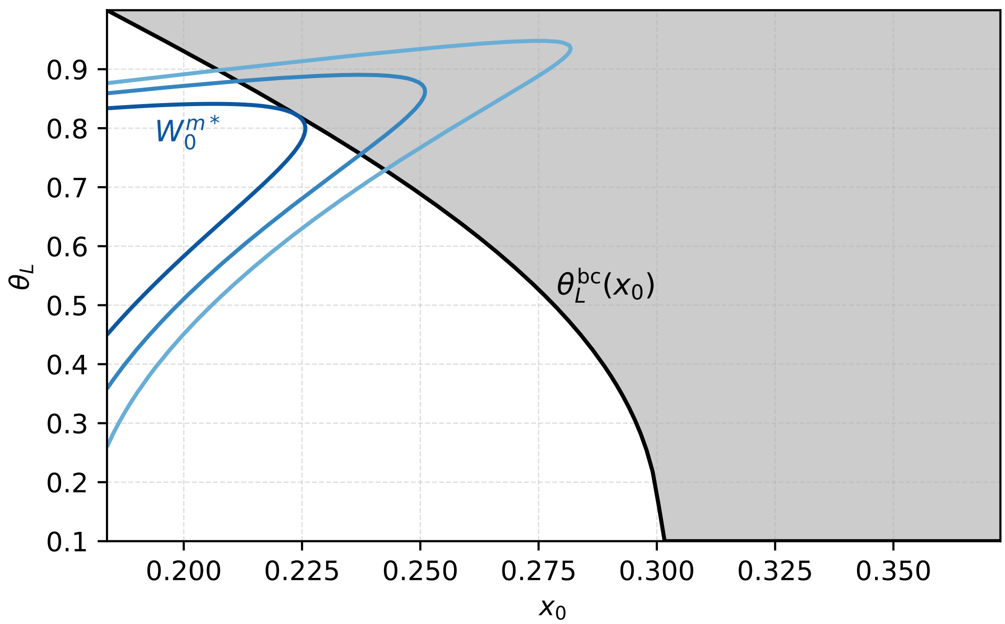


Figure 4: Indifference curves and determination of optimal pair $(x_0, \theta_L)$. We assume that $\theta \sim U[0.1, 1]$, $Y = 1$ and $c(q) = q^2$. The maximum pledgeable income is $I = 0.6 \times I^{\max}$ where $I^{\max} = \max_{x_1 \in [0,Y]} x_1 q(x_1)$. $W_0^{m*}$ corresponds to the value of the optimization problem in (26).

always lowers the borrower's payoff. By contrast, for a given $x_0$, the borrower's payoff is maximized when $\theta_L$ (and hence the debt limit) is neither too high nor too low. A high debt limit offers excessive financial flexibility and leads to over-borrowing, whereas a low limit restricts flexibility and results in under-borrowing.

The next proposition characterizes the optimal initial borrowing and its associated debt limit.

**Proposition 5** (Necessary Conditions Initial Debt)**.** *Let $x_0^{m*}$ be a solution to* (26)*, and $x_1^f(\theta_L^{m*})$ the corresponding debt limit that solves* (20)*.*

1. *If $x_1^*(\mathbb{E}[\theta]) \leq x_0^{\min}$ there is no flexibility for additional financing at $t = 1$. In this case, $x_0^{m*} = x_0^{\min}$ and $\theta_L^{m*} = \bar{\theta} = 1$.*

2. *If $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$ there is flexibility for additional financing at $t = 1$. In this case, $\theta_L^{m*} <$*

*$\bar{\theta} = 1$ and $x_0^{m*} > x_0^{\min}$. Specifically, $\{x_0^{m*}, \theta_L^{m*}\}$ jointly solve*

$$x_0^{m*}\mathbb{E}\left[q(x_1^f(\theta_L^{m*}, x_0^{m*}))\mathbf{1}_{\{\theta \leq \theta_L^{m*}\}} + q(x_1^f(\theta, x_0^{m*}))\mathbf{1}_{\{\theta > \theta_L^{m*}\}}\right] = I$$

$$x_0^{m*} = \frac{(1-\delta)\mathbb{E}\left[q(x_1^f(\theta_L^{m*}, x_0^{m*}))\mathbf{1}_{\{\theta \leq \theta_L^{m*}\}} + q(x_1^f(\theta_L^{m*}, x_0^{m*}))\mathbf{1}_{\{\theta > \theta_L^{m*}\}}\right]}{-\mathbb{E}\left[q'(x_1^f(\theta, x_0^{m*}))\frac{\partial x_1^f(\theta, x_0^{m*})}{\partial x_0}\mathbf{1}_{\{\theta > \theta_L^{m*}\}}\right]}$$

*and*

$$\delta = -\frac{x_0^{m*}}{\mathbb{E}[\theta_L^{m*} - \theta | \theta \leq \theta_L^{m*}]}\frac{q'(x_1^f(\theta_L^{m*}, x_0^{m*}))}{q(x_1^f(\theta_L^{m*}, x_0^{m*}))}.$$

Proposition 5 includes two cases based on the comparison between $x_1^*(\mathbb{E}[\theta])$ and $x_0^{\min}$. In the first case, $x_1^*(\mathbb{E}[\theta]) < x_0^{\min}$, so optimal total debt without knowing $\theta$ falls below the minimum initial debt required to satisfy the financing constraint. Therefore, the financing constraint by itself already causes over-borrowing, and the borrower's incentive to dilute old debt at $t = 1$ further exacerbates the problem. The optimal policy therefore imposes a debt limit that eliminates any further incentives of over-borrowing, which leads to no financial flexibility. In the second case, $x_1^*(\mathbb{E}[\theta]) \geq x_0^{\min}$, so the minimum initial debt remains below optimal total debt without knowing $\theta$. The optimal policy preserves some financial flexibility to balance between over- and under-borrowing. At $t = 1$, the agent can still borrow flexibly when the liquidity shock is low (high $\theta$).

A key feature of Proposition 5 is that the financing constraint always binds under non-exclusive lending. In other words, it is never optimal to borrow more than $I$ at $t = 0$. Instead, the optimal strategy selects initial debt $x_0$ and debt limit $\theta_L$ that provide exactly enough capital to fund investment $I$ and nothing more. The reason is straightforward: less borrowing at $t = 0$ reduces over-borrowing incentives at $t = 1$, creating greater financial flexibility.

Figure 5 compares the optimal contract under multiple lenders, $x_1^{m*}(\theta)$, with the single-lender solution $x_1^{s*}(\theta)$ and the second-best benchmark $x_1^*(\theta)$. In the left panel, which plots total debt, $x_1^{m*}(\theta)$ is flat for low $\theta$ and decreases monotonically once $\theta$ exceeds $\theta_L$. At low $\theta$, the borrower underborrows relative to both the second-best and single-lender solutions; at high $\theta$, she overborrows. Under the given parameters, the financing constraint binds under the single-lender case (i.e., $I > I^s$). As a result, $x_1^{s*}(\theta)$ exceeds $x_1^*(\theta)$ for $\theta < \theta_H$, becoming flat for $\theta > \theta_H$, and eventually falling below $x_1^*(\theta)$. The right panel shows the transfer the borrower receives at $t = 1$. In the second-best solution, transfers are normalized to zero. Under multiple lenders, transfers are constant for low $\theta$ and decline with $\theta$ thereafter. They fall below the single-lender case when $\theta$ is low but exceed it when $\theta$ is high.

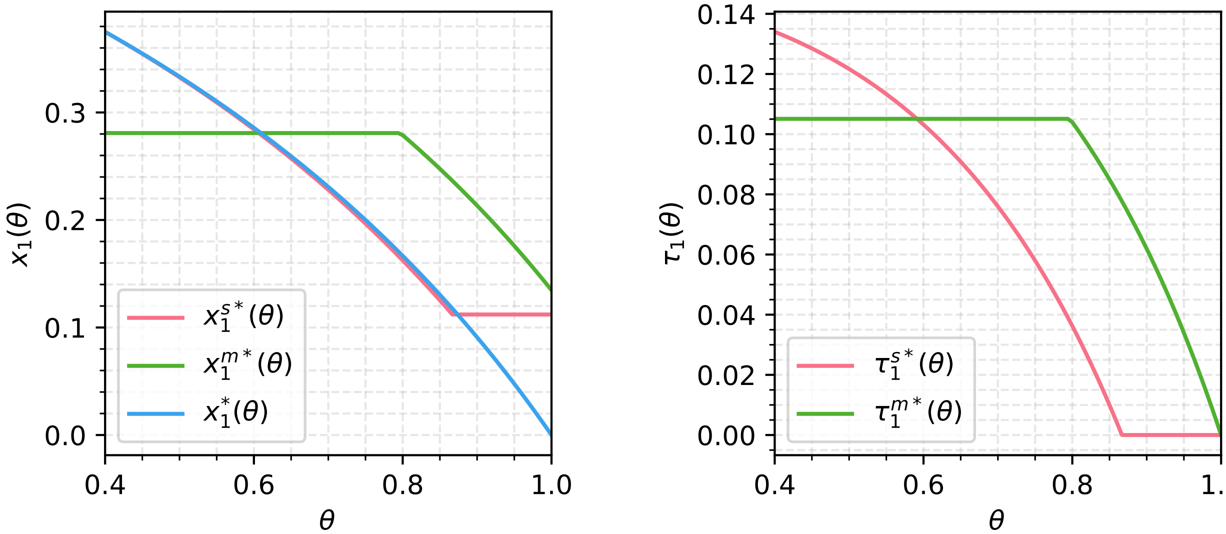


Figure 5: Optimal contract single lender and multiple lender. We assume that $\theta \sim U[0.4, 1]$, $Y = 1$, $c(q) = q^2$ and $I = 0.4 \times I^{\max}$, where the maximum pledgeable income is $I^{\max} = \max_{x_1 \in [0,Y]} \; x_1 q(x_1)$.

Figure 6 shows how the optimal debt policy under multiple lenders varies with investment size $I$. As $I$ increases, the initial debt level $x_0$ also rises, amplifying the borrower's incentive to dilute initial lenders at $t = 1$. This exacerbates over-borrowing, requiring a tighter debt limit. As a result, a larger range of types faces binding constraints ex-post. However, for types that remain unconstrained, the debt level increases with $I$.

The solution identified by Proposition 5 can be implemented using a widely observed contractual mechanism: an initial debt issuance $x_0^{m*}$ paired with a maintenance covenant that restricts total indebtedness to not exceed $x_1^f(\theta_L^{m*}, x_0^{m*})$. Such maintenance covenants are commonplace in corporate bond issuances and credit agreements. This arrangement restricts financial flexibility, as the borrower at $t = 1$ sometimes finds herself constrained from accessing her preferred level of additional debt when faced with financing needs. Meanwhile, this covenant creates a protection to existing creditors by restricting the dilution problem.

**Linear Example.** *The minimum feasible borrowing at $t = 0$ is*

$$x_0^{\min} = \frac{Y - \sqrt{Y^2 - 4cI}}{2}$$

*and*

$$x_1^*(\mathbb{E}[\theta]) = \frac{1 - \mathbb{E}[\theta]}{2 - \mathbb{E}[\theta]} Y.$$

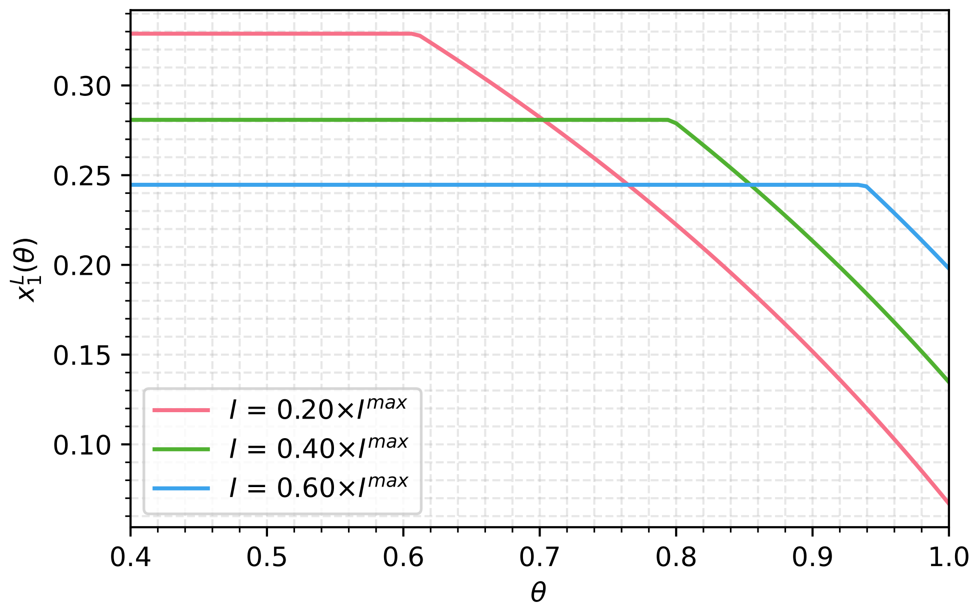


Figure 6: Comparative statics of face value at $t = 1$ for changes in initial investment $I$. We assume that $\theta \sim U[0.4, 1]$, $Y = 1$ and $c(q) = q^2$.

*From here, we get that* $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$ *if* $\mathbb{E}[\theta] < \frac{2\sqrt{Y^2-4cI}}{Y+\sqrt{Y^2-4cI}}$. *In this case,* $(x_0, \theta_L)$ *solve*

$$I = \frac{x_0(Y - x_0)}{c}\mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta\leq\theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right]$$

$$\frac{x_0}{Y - x_0} = \frac{(1-\delta)\mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta\leq\theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right]}{\mathbb{E}\left[\frac{1}{2-\theta}\mathbf{1}_{\{\theta>\theta_L\}}\right]}$$

*where*

$$\delta = \frac{2-\theta_L}{\mathbb{E}[\theta_L - \theta|\theta \leq \theta_L]}\frac{x_0}{Y - x_0}.$$

In the linear example, we have the following comparative statics result.

**Proposition 6.** *Consider the linear example and define*

$$\xi \equiv \frac{c \cdot I}{Y}.$$

*The threshold* $\theta_L$ *and debt ratio* $\frac{x_0}{Y}$ *that solve the equations in Proposition 5 depend on* $(c, I, Y)$ *only through the composite parameter* $\xi$. *Additional financing flexibility at* $t = 1$ *(i.e.,* $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$*)*

*exists if*

$$\mathbb{E}[\theta] < \frac{2\sqrt{1-4\xi}}{1+\sqrt{1-4\xi}}.$$

*In this case, both $\theta_L$ and $\frac{x_0}{Y}$ are increasing in $\xi$. Specifically, they increase with $c$ and $I$ but decrease with $Y$.*

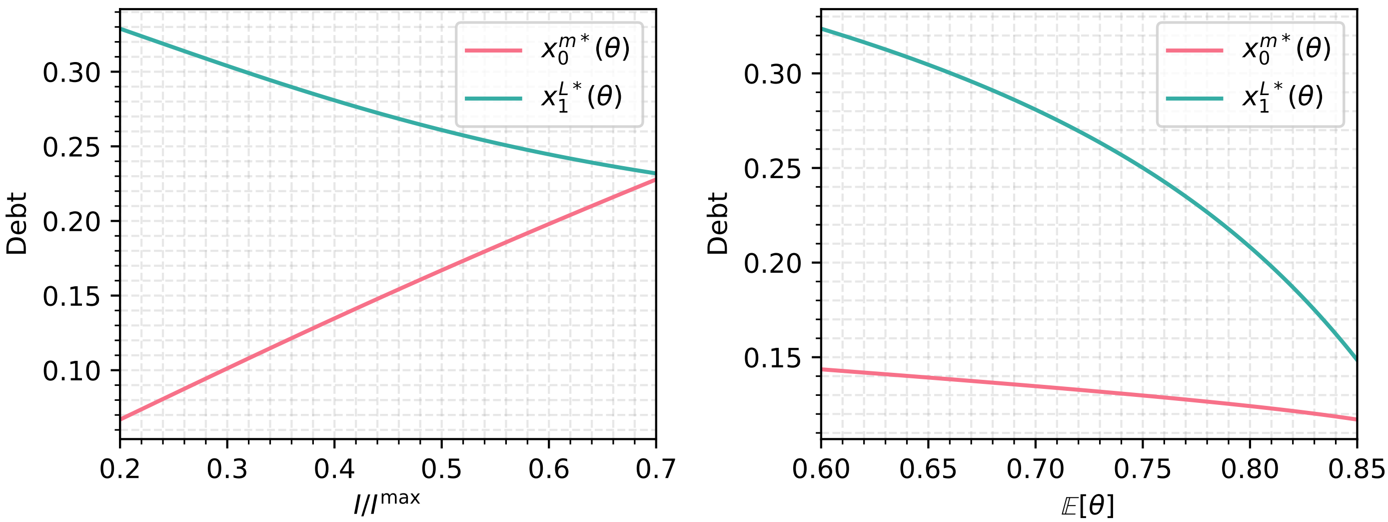


Figure 7: Comparative statics debt limit. We assume $Y = 1$, $c(q) = q^2$, and $G(\theta) = \left(\frac{\theta-\underline{\theta}}{1-\underline{\theta}}\right)^{\gamma}$, where $\underline{\theta} = 0.4$. The expected liquidity shock is given by $\mathbb{E}[\theta] = \underline{\theta} + \frac{\gamma}{\gamma+1}(1-\underline{\theta})$. In the left panel, we consider the uniform case with $\gamma = 1$ and consider $I/I^{\max} \in [0.2, 0.7]$. In the right panel, we take $I/I^{\max} = 0.4$ and consider $\gamma \in [0.5, 3]$.

We plot these results in Figure 7. The left panel shows how the optimal solution under multiple lenders varies with investment size $I$, normalized by the maximum pledgeable income $I^{\max}$. As $I$ increases, the borrower must raise more initial funds, leading to a higher $x_0^{m*}$. To mitigate stronger incentives to overborrow at $t = 1$, the optimal debt limit $x_1^{L*}$ must decrease. The right panel shows how the multiple lender solution varies with $\mathbb{E}[\theta]$. We increase $\mathbb{E}[\theta]$ by shifting the distribution according to first-order stochastic dominance. As $\mathbb{E}[\theta]$ increases, there is less need to borrow at $t = 1$, and the value of financial flexibility becomes less important. As a result, over-borrowing becomes a relatively bigger concern. Consequently, the optimal debt limit $x_1^{L*}$ decreases to provide tighter borrowing constraints. Since initial lenders face lower dilution, they are willing to lend more for any given $x_0$. To raise the required investment $I$, the optimal initial debt $x_0^{m*}$ can therefore be lower.

Next, we examine the effect of uncertainty by examining the impact of increasing the dispersion in the distribution of liquidity shocks $\theta$. In the following example, we model the liquidity shocks $\theta$ using a beta distribution, because this allows us to increase the dispersion while keeping a constant mean without changing the support $[\underline{\theta}, \bar{\theta}]$. By varying the shape parameters appropriately, we can adjust the dispersion of $\theta$ without altering its mean or its bounded support. In particular, we set $\bar{\theta} = 1$ and let

$$g(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\theta - \underline{\theta}}{1-\underline{\theta}}\right)^{\alpha-1} \left(\frac{1-\theta}{1-\underline{\theta}}\right)^{\beta-1} \frac{1}{1-\underline{\theta}}, \ \forall \theta \in [\underline{\theta}, 1],$$

where $\Gamma(\cdot)$ denotes the gamma function and

$$\alpha = k\frac{\mathbb{E}[\theta] - \underline{\theta}}{1-\underline{\theta}}$$
$$\beta = k\frac{1 - \mathbb{E}[\theta]}{1-\underline{\theta}}$$

The spread factor $k$ measures the dispersion of the distribution and is inversely related to the variance of the distribution. In particular, the standard deviation of $\theta$ is

$$\sigma(\theta) = \sqrt{\frac{(1-\mathbb{E}[\theta])(\mathbb{E}[\theta] - \underline{\theta})}{k+1}}.$$

Figure 8 shows that the debt limit gets higher as $\sigma$ increases because the value of financial flexibility increases with the uncertainty in liquidity shock.

# 5 Extensions

## 5.1 Short-Term Debt

So far, we have only allowed initial lenders to hold long-term debt due at $t = 2$ to highlight the dilution and non-exclusivity problem. However, short-term debt is another natural solution. We now explore this possibility by allowing the borrower to finance the upfront investment at $t = 0$ using a combination of short-term debt $b$ (maturing at $t = 1$) and long-term debt $x_0$ (maturing at $t = 2$). Since the borrower is financially constrained at $t = 1$, she must issue new debt to repay the short-term debt $b$ due at that time.

It is immediately clear that short-term debt must be riskless because there is no cash-flow risk at $t = 1$.[8] In fact, because short-term debt is riskless, it completely crowds out long-term debt. We have the following result.

[8] If the borrower were to default under some report $\hat{\theta}$ (resulting in zero payoff), she would instead choose a report $\hat{\theta}'$ that avoids default and yields a positive payoff. Thus, truthful reporting requires that default never occurs, meaning short-term debt must be riskless.

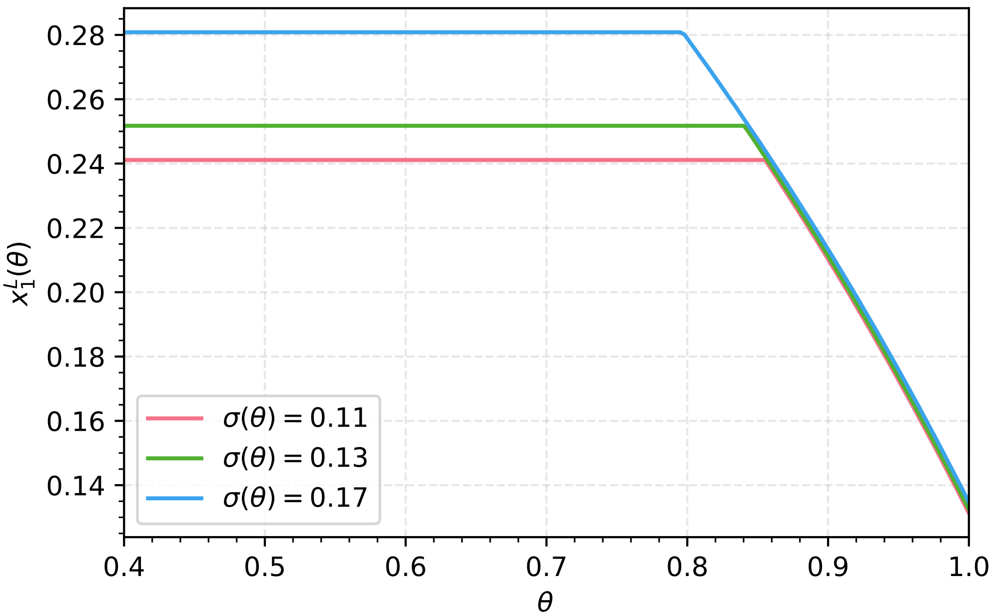


Figure 8: Effect of mean-preserving spread in liquidity shocks. We assume $Y = 1$, $c(q) = q^2$, and $G(\theta)$ is beta distributed with $\mathbb{E}[\theta] = 0.7$ and $\Theta = [0.4, 1]$.

**Proposition 7.** *If the borrower can issue both short- and long-term debt, she will exclusively borrow short-term debt $b = I$ due at $t = 1$. At $t = 1$, she rolls over the short-term debt by borrowing $x_1(\theta) = \max x_1(\theta), x_1(\theta_H)$ from new lenders, where $\theta_H$ satisfies $b = x_1(\theta_H)q(x_1(\theta_H))$.*

In practice, firms rarely issue only short-term debt because our analysis has abstracted from its potential downsides. For instance, our model assumes frictionless debt markets, thereby ignoring any costs associated with rolling over short-term debt. Moreover, results would differ if we introduced cash-flow uncertainty at $t = 1$. In a related setting with such uncertainty and without commitment to debt issuance policies, Hu et al. (2021) show that borrowers may issue long-term debt when short-term debt becomes risky. This also relates to Diamond and He (2014), who find that short-term debt can create greater debt overhang than long-term debt. Thus, our finding that short-term debt completely crowds out long-term debt results from omitting these frictions and should be interpreted cautiously. We emphasize, however, that debt maturity is not this paper's central focus.

While short-term debt dominates the long-term debt solution with a debt limit studied earlier, it still fails to implement the constrained optimal allocation. This occurs because repayment at $t = 1$ is not contingent on the borrower's liquidity shock. The second-best solution allows the borrower to pledge more cash flow when $\theta$ is low and less when $\theta$ is high. In the next subsection, we show how long-term debt with acceleration clauses can achieve this allocation.

### 5.2 Mandatory Prepayment

An alternative to debt limits for curbing excessive new debt issuance is a mandatory prepayment covenant. Debt sweeps are common in practice, and they are prepayment provisions that require borrowers to use part of new debt proceeds to repay existing debt.

Suppose when the borrower reports $\hat{\theta}$, an amount $b(\hat{\theta}) \in [0, x_0]$ becomes due at $t = 1$. Under truthful reporting ($\hat{\theta} = \theta$), the net proceeds from new debt issuance after prepayment are:

$$\tau_1(\theta) = (x_1(\theta) - x_0)q(x_1(\theta) - b(\theta)) - b(\theta).$$

Anticipating the mandatory prepayment provision, lenders' participation constraint at $t = 0$ becomes

$$\tau_0 = \mathbb{E}\left[(x_0 - b(\theta))q(x_1(\theta) - b(\theta)) + b(\theta)\right].$$

Let $x_1^+(\theta) \equiv x_1(\theta) - b(\theta)$ be the face value of debt *after* the prepayment and let $m_1(\theta) \equiv b(\theta)\left(1 - q(x_1^+(\theta))\right)$, we can write

$$\tau_1(\theta) = (x_1^+(\theta) - x_0)q(x_1^+(\theta)) - m_1(\theta),$$

where $m_1(\theta)$ corresponds to the transfer of funds from new lenders to old lenders. The participation constraint for the initial borrowers implies that

$$\tau_0 = \mathbb{E}\left[x_0 q(x_1^+(\theta)) + m_1(\theta)\right].$$

Substituting in the borrower's expected payoff at $t = 0$, we get

$$\tau_0 - I + \mathbb{E}\Big[\tau_1(\theta) + \theta V_2\left(x_1^+(\theta)\right)\Big] = \mathbb{E}\left[W_1(x_1^+(\theta), \theta)\right] - I.$$

If the borrower is financially constrained, we need to have that $\tau_1(\theta) \geq 0$, which requires that

$$(x_1^+(\theta) - x_0)q(x_1^+(\theta)) \geq m_1(\theta).$$

We can write the borrower problem at $t = 0$ as

$$
\begin{aligned}
&\max_{x_0, x_1^+(\theta), m_1(\theta)} \mathbb{E}\left[W_1(x_1^+(\theta), \theta)\right] - I \\
&\text{subject to} \\
&\theta \in \arg\max_{\hat{\theta}} \left\{V_1(x_1^+(\hat{\theta}), x_0, \theta) - m_1(\hat{\theta})\right\} \\
&\mathbb{E}\left[x_0 q(x_1^+(\theta)) + m_1(\theta)\right] \geq I \\
&m_1(\theta) \leq (x_1^+(\theta) - x_0) q(x_1^+(\theta)) \\
&m_1(\theta) \leq x_0 \left(1 - q(x_1^+(\theta))\right).
\end{aligned}
\tag{27}
$$

We have the following result.

**Proposition 8.** *Let $x_1^{s*}(\theta)$ be the solution to the single lender problem in Proposition 2. If*

$$
W_1(x_1^{s*}(\underline{\theta}), \underline{\theta}) - \theta_H V_2(x_1^{s*}(\theta_H)) + \int_{\underline{\theta}}^{\theta_H} V_2(x_1^{s*}(\tilde{\theta})) d\tilde{\theta} \leq x_1^{s*}(\theta_H) \tag{28}
$$

*then $x_0 = x_1^{s*}(\bar{\theta})$ and $x_1^{+*}(\theta) = x_1^{s*}(\theta)$ together with*

$$
m_1(\theta) = V_1(x_1^{+*}(\theta), x_0, \theta) - \theta_H V_2(x_1^{+*}(\theta_H)) + \int_{\theta}^{\theta_H} V_2(x_1^{+*}(\tilde{\theta})) d\tilde{\theta}
$$

*solve the optimal prepayment problem (27).*

**Implementation** Next, we look at the implementation of the mandatory prepayment. First, we let

$$
x_1(\theta) = x_1^+(\theta) + b(\theta) = x_1^+(\theta) + \frac{m_1(\theta)}{1 - q(x_1^+(\theta))}
$$

If $x_1(\theta)$ is decreasing on $[\underline{\theta}, \theta_H]$, we can define the inverse $\theta(x_1) = x_1^{-1}(\theta)$. In this case, we can write the prepayment as a function of the issuance $\Delta x_1$ at $t = 1$ as

$$
B(\Delta x_1) = b(\theta(x_0 + \Delta x_1)).
$$

In terms of implementation, we can write the prepayment as a debt sweep specifying the percentage of the amount issued that needs to be used to repay debt. In particular, letting $B(\Delta x_1) = \alpha(\Delta x_1) q(x_0 + \Delta x_1) \Delta x_1$ we can specify a non-linear debt sweep

$$
\alpha(\Delta x_1) = \frac{B(\Delta x_1)}{q(x_0 + \Delta x_1) \Delta x_1}.
$$

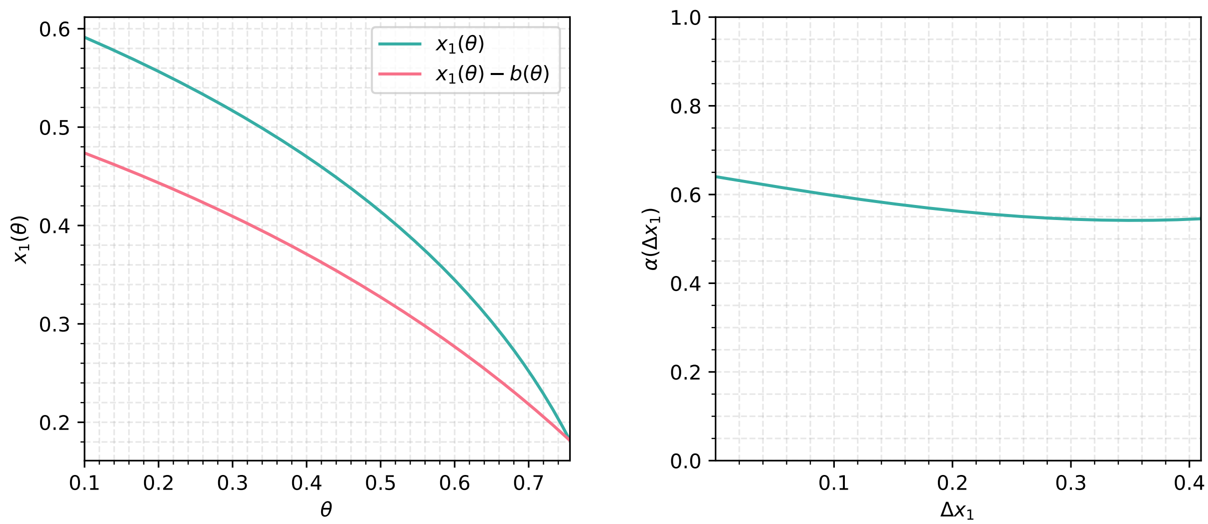


Figure 9: Implementation of optimal contract using non-linear debt sweep. We assume that $\theta \sim U[0.4, 1]$, $Y = 1$, $c(q) = q^2$, and $I = 0.4 \times I^{\max}$.

Figure 9 illustrates an example of a non-linear debt sweep in the context of the linear example.

# 6 Conclusion

This paper examines the optimal design of financial flexibility for firms facing financing constraints and non-exclusive lending. We show that the tradeoff between preserving access to liquidity and preventing excessive borrowing leads to an endogenous debt limit as the optimal contractual solution. This limit allows firms to borrow flexibly when facing modest liquidity shocks while constraining over-borrowing during severe shocks. Our model provides a theoretical foundation for the prevalence of maintenance covenants in debt contracts, demonstrating how they mitigate creditor dilution while preserving necessary financial flexibility. The analysis highlights how financing constraints shape firms' borrowing capacity and offers insights into the trade-offs involved in designing optimal debt contracts under asymmetric information and non-exclusive lending. Future research could explore how stochastic cash flows or alternative covenant structures might further refine these contractual solutions.

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# Appendix

## A Single Lender Equilibrium

For notational convenience, let us define

$$\bar{V}_1 = \tau_1(\bar{\theta}) + \bar{\theta} V_2(x_1(\bar{\theta}))$$

as type $\bar{\theta}$'s continuation payoff at $t = 1$.

**Lemma 3.** *Any incentive-compatible pair $\{\tau_1(\theta), x_1(\theta)\}$ satisfying $\bar{V}_1 \geq \bar{\theta} V_2(x_1(\bar{\theta}))$ satisfies the constraint*

$$\tau_1(\theta) = \bar{V}_1 - \theta V_2\left(x_1(\theta)\right) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \geq 0 \quad \forall \theta \in \Theta.$$

**Proof of Lemma 3**

*Proof.* We verify that it is sufficient to consider the non-negativity constraint of $\tau_1(\theta)$ for the highest type. Integration by parts implies that

$$\theta V_2\left(x_1(\theta)\right) + \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} = \bar{\theta} V_2\left(x_1(\bar{\theta})\right) - \int_{\theta}^{\bar{\theta}} \tilde{\theta} V_2'(x_1(\tilde{\theta}))dx_1(\tilde{\theta})$$

Hence, we can write the envelope condition as

$$\tau_1(\theta) = \tau_1(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} \tilde{\theta} V_2'(x_1(\tilde{\theta}))dx_1(\tilde{\theta})$$

It follows that, for any $\theta'' > \theta'$, we have

$$\tau_1(\theta'') = \tau_1(\theta') - \int_{\theta'}^{\theta''} \tilde{\theta} \underbrace{V_2'(x_1(\tilde{\theta}))}_{<0} \underbrace{dx_1(\tilde{\theta})}_{\leq 0} \leq \tau_1(\theta'),$$

which means that $\tau_1(\bar{\theta}) \geq 0$ implies $\tau_1(\theta) \geq 0$ for all $\theta < \bar{\theta}$. Moreover, the constraint $\tau(\bar{\theta}) \geq 0$ is equivalent to require that $\bar{V}_1 \geq \bar{\theta} V_2\left(x_1(\bar{\theta})\right)$ □

**Proof of Proposition 1**

*Proof.* Let $\mathbf{x}_1 \equiv \{x_1(\theta)\}_{\theta\in\Theta}$. For a fixed $\lambda$ (and therefore a fixed $\delta$) we have

$$\max_{\mathbf{x}_1} \mathcal{U}(\mathbf{x}_1) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \left[W_1(x_1(\theta),\theta) + (1-\delta)\frac{G(\theta)}{g(\theta)} V_2\left(x_1(\theta)\right)\right] dG(\theta) - (1-\delta)\bar{\theta}V_2\left(x_1(\bar{\theta})\right)$$

s.t. $\mathbf{x}_1$ is non-increasing.

Starting from $\mathbf{x}_1$, the directional derivative in the direction $h(\theta)$ is generally defined as

$$\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{h}) = \frac{d}{dk}\mathcal{U}(\mathbf{x}+k\mathbf{h})\Big|_{k=0},$$

which, in our case, is given by

$$\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{h}) = (1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)h(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \left[\frac{\partial W_1(x_1(\theta),\theta)}{\partial x_1} g(\theta) - (1-\delta)G(\theta)q\left(x_1(\theta)\right)\right] h(\theta)d\theta.$$

Letting

$$\Phi(\theta;\mathbf{x}_1) \equiv \int_{\underline{\theta}}^{\theta} \left[\frac{\partial W_1(x_1(\tilde{\theta}),\tilde{\theta})}{\partial x_1} g(\tilde{\theta}) - (1-\delta)G(\tilde{\theta})q\left(x_1(\tilde{\theta})\right)\right] d\tilde{\theta},$$

the directional derivative can be written as

$$\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{h}) = (1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)h(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \Phi'(\theta;\mathbf{x}_1)h(\theta)d\theta.$$

Using integration by parts, we obtain that $\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{h})$ can be written as

$$\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{h}) = \left[\Phi(\bar{\theta};\mathbf{x}_1) + (1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)\right]h(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta;\mathbf{x}_1)dh(\theta).$$

This alternative formulation for the directional derivative is convenient because it is written explicitly in terms of $dh(\theta)$ rather than $h(\theta)$. This allows us to incorporate the monotonicity constraint directly into the analysis of the first-order conditions.

Let $P$ be set of all non-increasing functions on $\Theta$. Lemma 1 in (Luenberger, 1969, p. 227) provides the following necessary and sufficient condition for $\mathbf{x}_1$ to maximize $\mathcal{U}$:

$$\nabla\mathcal{U}(\mathbf{x}_1;\tilde{\mathbf{x}}_1) \leq 0 \quad \forall\ \tilde{\mathbf{x}}_1 \in P$$
$$\nabla\mathcal{U}(\mathbf{x}_1;\mathbf{x}_1) = 0.$$

In our case, these two conditions become

$$\left[\Phi(\bar{\theta};\mathbf{x}_1)+(1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)\right]\tilde{x}_1(\bar{\theta})-\int_{\underline{\theta}}^{\bar{\theta}}\Phi(\theta;\mathbf{x}_1)d\tilde{x}_1(\theta)\leq 0\quad\forall\ \tilde{\mathbf{x}}_1\in P \tag{29a}$$

$$\left[\Phi(\bar{\theta};\mathbf{x}_1)+(1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)\right]x_1(\bar{\theta})-\int_{\underline{\theta}}^{\bar{\theta}}\Phi(\theta;\mathbf{x}_1)dx_1(\theta)=0. \tag{29b}$$

We are going to construct a solution, $\mathbf{x}_1$, given by

$$x_1(\theta)=\begin{cases}x_1^s(\theta) & \text{if } \theta\in[\underline{\theta},\theta_H)\\ x_1^s(\theta_H) & \text{if } \theta\in[\theta_H,\bar{\theta}]\end{cases}$$

and show that it satisfies (29). Note (14) implies that $x_1^s(\theta)$ solves

$$\frac{\partial W_1(x_1,\theta)}{\partial x_1}-(1-\delta)\frac{G(\theta)}{g(\theta)}q\left(x_1\right)=0.$$

Therefore, $\Phi(\theta;\mathbf{x}_1)=0$ for all $\theta\in[\underline{\theta},\theta_H]$. Moreover, we define $\theta_H\in[\underline{\theta},\bar{\theta}]$ as the solution to

$$\int_{\theta_H}^{\bar{\theta}}\left[\frac{\partial W_1(x_1^s(\theta_H),\tilde{\theta})}{\partial x_1}g(\tilde{\theta})-(1-\delta)q\left(x_1^s(\theta_H)\right)G(\tilde{\theta})\right]d\tilde{\theta}+(1-\delta)\bar{\theta}q\left(x_1^s(\theta_H)\right)=0. \tag{30}$$

We will show at the end of this proof that this solution exists. By construction, we have

$$\Phi(\bar{\theta};\mathbf{x}_1)+(1-\delta)\bar{\theta}q\left(x_1(\bar{\theta})\right)=0,$$

and (29b) holds. Moreover, (29a) becomes

$$-\int_{\underline{\theta}}^{\bar{\theta}}\Phi(\theta;\mathbf{x}_1)d\tilde{x}_1(\theta)\leq 0\quad\forall\ \tilde{\mathbf{x}}_1\in P.$$

Assumptions 4 and $x_1^s(\theta)$ is non-increasing in $\theta$ imply that

$$\frac{\partial W_1(x_1,\theta)}{\partial x_1}\Big|_{x_1=x_1^s(\theta_H)}-(1-\delta)\frac{G(\theta)}{g(\theta)}q\left(x_1^s(\theta_H)\right)\leq 0,\quad\forall\theta\in[\theta_H,\bar{\theta}]$$

so $\Phi(\theta;\mathbf{x}_1)\leq 0$ for all $\theta\in(\theta_H,\bar{\theta}]$. Therefore, the first order condition (29) reduces to

$$\int_{\theta_H}^{\bar{\theta}}\Phi(\theta;\mathbf{x}_1)d\tilde{x}_1(\theta)\geq 0\quad\forall\ \tilde{\mathbf{x}}_1\in P,$$

which is satisfied for all non-increasing $\tilde{\mathbf{x}}_1$.

The final step is to consider the existence of a solution to equation (30). Note that

$$\frac{\partial^2 W_1(x_1^s(\theta_H),\theta)}{\partial\theta\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial W_1(x_1^s(\theta_H),\theta)}{\partial\theta}\right) = \frac{\partial V_2(x_1^s(\theta_H))}{\partial x_1} = -q(x_1^s(\theta_H)).$$

Therefore, for $\theta \in [\theta_H, \bar{\theta}]$, we have

$$\frac{\partial W_1(x_1^s(\theta_H),\theta)}{\partial x_1} = \frac{\partial W_1(x_1^s(\theta_H),\theta_H)}{\partial x_1} + (\theta_H - \theta)q(x_1(\theta_H)) = (\theta_H - \theta)q(x_1(\theta_H)) + (1-\delta)\frac{G(\theta_H)}{g(\theta_H)}q\left(x_1(\theta_H)\right),$$

which allow us to simplify equation (30) in the following way

$$q(x_1^s(\theta_H))\int_{\theta_H}^{\bar{\theta}}\left[(\theta_H - \tilde{\theta}) + (1-\delta)\frac{G(\theta_H)}{g(\theta_H)} - (1-\delta)\frac{G(\tilde{\theta})}{g(\tilde{\theta})}\right]g(\tilde{\theta})d\tilde{\theta} + (1-\delta)\bar{\theta}q\left(x_1^s(\theta_H)\right) = 0.$$

Hence, we can cancel $q(x_1^s(\theta_H))$ and obtain an expression exclusively in terms of $\delta$ and $g(\theta)$

$$(1-\delta)\frac{G(\theta_H)(1-G(\theta_H))}{g(\theta_H)} + \int_{\theta_H}^{\bar{\theta}}\left[(\theta_H - \tilde{\theta})g(\tilde{\theta}) - (1-\delta)G(\tilde{\theta})\right]d\tilde{\theta} + (1-\delta)\bar{\theta} = 0.$$

Integrating by parts, we get

$$\int_{\theta_H}^{\bar{\theta}}\left[(\theta_H - \tilde{\theta})g(\tilde{\theta}) - G(\tilde{\theta})\right]d\tilde{\theta} = \theta_H - \bar{\theta}$$

so, we end with

$$H(\theta_H) \equiv (1-\delta)\frac{G(\theta_H)(1-G(\theta_H))}{g(\theta_H)} + \theta_H - \delta\left[\bar{\theta} - \int_{\theta_H}^{\bar{\theta}} G(\tilde{\theta})d\tilde{\theta}\right] = 0.$$

The function $H(\theta_H)$ satisfies the following properties

$$\begin{aligned}
H(\underline{\theta}) &= \underline{\theta} - \delta\mathbb{E}[\theta] \\
H(\bar{\theta}) &= (1-\delta)\bar{\theta} \\
H'(\theta_H) &= \left[\frac{(2-\delta G(\theta_H))}{1-G(\theta_H)}\frac{g(\theta_H)}{G(\theta_H)} - (1-\delta)\frac{g'(\theta_H)}{g(\theta_H)}\right]\frac{G(\theta_H)(1-G(\theta_H))}{g(\theta_H)}
\end{aligned}$$

Note that we can write

$$H'(\theta_H) = \left[ \frac{1+\delta+G(\theta_H)}{1-G(\theta_H)} \frac{g(\theta_H)}{G(\theta_H)} + (1-\delta) \frac{\partial \log \frac{G(\theta_H)}{g(\theta_H)}}{\partial \theta_H} \right] \frac{G(\theta_H)(1-G(\theta_H))}{g(\theta_H)}$$

If $G(\theta)$ is log-concave, then $G(\theta)/g(\theta)$ is increasing, which means that $\frac{\partial \log \frac{G(\theta_H)}{g(\theta_H)}}{\partial \theta_H} > 0$. So $H'(\theta_H) > 0$. Hence, a solution to the equation (30) exists only if $\delta \geq \underline{\theta}/\mathbb{E}[\theta]$.

If $\delta < \underline{\theta}/\mathbb{E}[\theta]$, the first order condition is satisfied by setting $\theta_H = \underline{\theta}$ and $x_1(\theta) = x_1^p$ such that

$$\Phi(\bar{\theta}; \mathbf{x}_1 = x_1^p) + (1-\delta)\bar{\theta} q\left(x_1^p\right) = 0,$$

which requires that

$$\int_{\underline{\theta}}^{\bar{\theta}} \Big[ \frac{\partial W_1(x_1^p, \tilde{\theta})}{\partial x_1} g(\tilde{\theta}) - (1-\delta) q(x_1^p) G(\tilde{\theta}) \Big] d\tilde{\theta} + (1-\delta)\bar{\theta} q(x_1^p) = 0.$$

Using integration by parts, we get that

$$x_1^p q'(x_1^p) + \left(1 - \delta \mathbb{E}[\theta]\right) q(x_1^p) = 0.$$

Noticing that $x_1^s(\underline{\theta})$ solves

$$x_1^s(\underline{\theta}) q'(x_1^s(\underline{\theta})) + \left(1 - \underline{\theta}\right) q(x_1^s(\underline{\theta})) = 0$$

we realize that whenever $\underline{\theta} > \delta \mathbb{E}[\theta]$, we have that $x_1^p > x_1^s(\underline{\theta})$. This, in turn implies that

$$\frac{\partial W_1(x_1^p, \theta)}{\partial x_1} g(\theta) - (1-\delta) q(x_1^p) G(\theta) < 0$$

so $\Phi(\theta; \mathbf{x}_1 = x_1^p) < 0$ which means that

$$-\int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta; \mathbf{x}_1 = x_1^p) d\tilde{x}_1(\theta) \leq 0 \quad \forall\, \tilde{\mathbf{x}}_1 \in P,$$

which means that $x_1(\theta) = x_1^p$ is optimal. $\square$

**Proof of Proposition 2**

Suppose that $\delta = 1$ (or equivalently $\lambda = 0$). In this case, it can be readily verified that $\theta_H = \bar{\theta}$ and $x_1^e(\theta) = x_1^*(\theta)$. This candidate solution – together with the transfer at $t = 1$ defined by (10) –

solves the single lender equilibrium under incomplete information only if the financing constraint is satisfied, that is

$$\tau_0 = \mathbb{E}\left[W_1\left(x_1^*(\theta), \theta\right) + \frac{G(\theta)}{g(\theta)} V_2\left(x_1^*(\theta)\right)\right] - \bar{\theta} V_2\left(x_1^*(\bar{\theta})\right) = \bar{I}^e \geq I$$

Suppose this is not the case, and assume that

$$\mathbb{E}\left[W_1\left(x_1^*(\theta), \theta\right) + V_2\left(x_1^*(\theta)\right) \frac{G(\theta)}{g(\theta)}\right] - \bar{\theta} V_2\left(x_1^*(\bar{\theta})\right) < I,$$

We show that there exists $\delta \in [0,1]$ such that

$$\mathcal{T}_0(\delta) = \mathbb{E}\left[W_1\left(x_1^e(\theta,\delta), \theta\right) + V_2\left(x_1^e(\theta,\delta)\right) \frac{G(\theta)}{g(\theta)}\right] - \bar{\theta} V_2\left(x_1^e(\bar{\theta},\delta)\right) = I$$

First, notice that when $\delta = 1$, we have that $x_1^e(\theta,\delta) = x_1^*(\theta)$. Hence, from our initial hypothesis, $\mathcal{T}_0(1) < I$. Next, consider the case when $\delta = 0$. In this case, we have that $\theta_H = \underline{\theta}$ and $x_1^e(\theta,\delta) = x_1^p$ where

$$x_1^p q'(x_1^p) + q(x_1^p) = 0$$

which means that $x_1^p = \arg\max_x xq(x)$. It follows then that

$$\mathcal{T}_0(0) = x_1^p q(x_1^p) + \mathbb{E}\left[\theta + \frac{G(\theta)}{g(\theta)}\right] V_2\left(x_1^p\right) - \bar{\theta} V_2\left(x_1^p\right) = x_1^p q(x_1^p).$$

By Assumption 2, this means that $\mathcal{T}_0(0) > I$. By the maximum theorem, $x_1^e(\theta,\delta)$ is continuous in $\delta$, and so it is $\mathcal{T}_0(\delta)$. This means that there exists some $\delta^e \in [0,1]$ such that $\mathcal{T}_0(\delta^e) = I$.

# B Multiple Lender Equilibrium

Let us first rewrite the problem in (19). Substituting $W_1(x_1,\theta) = x_0 q(x_1) + V_1(x_1, x_0, \theta)$ together with equation (18) into the objective function in (19) and applying integration by parts, we obtain the following problem:

$$\begin{aligned}
W_0(x_0) = \max_{x_1(\theta)\geq x_0} \; & V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E}\left[x_0 q(x_1(\theta)) - \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta))\right] \\
& \text{subject to} \\
& V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta}))d\tilde{\theta} \quad \forall\theta \in \Theta \\
& x_0 \mathbb{E}\left[q(x_1(\theta))\right] \geq I \\
& x_1(\theta) \text{ is non-increasing.}
\end{aligned} \tag{31}$$

**Proof of Proposition 3**

*Proof.* Letting $\lambda$ be the multiplier of the investment constraint, we write

$$\max_{\theta_L} \int_{\underline{\theta}}^{\theta_L} \left(W_1(x_1^f(\theta_L), \theta) + \lambda x_0 q(x_1^f(\theta_L))\right) dG(\theta) + \int_{\theta_L}^{\bar{\theta}} \left(W_1(x_1^f(\theta), \theta) + \lambda x_0 q(x_1^f(\theta))\right) dG(\theta).$$

Dividing by $1+\lambda$ and defining $\delta \equiv 1/(1+\lambda)$, we can rewrite the previous optimization problem in the following equivalent form

$$\max_{\theta_L} \int_{\underline{\theta}}^{\theta_L} \left((1-\delta)x_0 q(x_1^f(\theta_L)) + \delta W_1(x_1^f(\theta_L), \theta)\right) dG(\theta) + \int_{\theta_L}^{\bar{\theta}} \left((1-\delta)x_0 q(x_1^f(\theta)) + \delta W_1(x_1^f(\theta), \theta)\right) dG(\theta).$$

Next, we consider the first-order condition for $\theta_L$, which is given by

$$\begin{aligned}
\frac{\partial \text{ObjFun}}{\partial \theta_L} &= x_1^{f\prime}(\theta_L) \int_{\underline{\theta}}^{\theta_L} \left(x_0 q'(x_1^f(\theta_L)) + \delta(\theta_L - \theta) q(x_1^f(\theta_L))\right) dG(\theta) \\
&= x_1^{f\prime}(\theta_L) G(\theta_L) q(x_1^f(\theta_L)) \left\{ x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta \mathbb{E}[\theta_L - \theta | \theta \leq \theta_L] \right\} = 0
\end{aligned}$$

Assumptions 3 and 1 imply that $G(\theta)$ and $q(x)$ are log-concave; accordingly, the functions $E[\theta_L - \theta|\theta \leq \theta_L]$ and $\frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))}$ are increasing in $\theta_L$. It follows that the objective function is quasi-concave, so any local maximum is also a global one. The second-order condition, evaluated at a local maximum $\theta_L$, is

$$\frac{\partial^2 \text{ObjFun}}{\partial \theta_L^2} = x_1^{f\prime}(\theta_L) \left[ \delta G(\theta_L) + x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} g(\theta_L) \right] q(x_1^f(\theta_L)) + \\ + \left( x_1^{f\prime}(\theta_L) \right)^2 G(\theta_L) \left[ x_0 q''(x_1^f(\theta_L)) + \delta \mathbb{E}[\theta_L - \theta | \theta \leq \theta_L] q'(x_1^f(\theta_L)) \right].$$

First, we verify that $\theta_L > \underline{\theta}$. Ealuting the second order condition at at $\theta_L = \underline{\theta}$, we get that $\frac{\partial \text{ObjFun}}{\partial \theta_L} = 0$ and $\frac{\partial^2 \text{ObjFun}}{\partial \theta_L^2} > 0$, which means that $\theta_L = \underline{\theta}$ is a local minimum. Second, the solution is interior is interiero (that is, $\theta_L < \bar{\theta}$) only if $\left.\frac{\partial \text{ObjFun}}{\partial \theta_L}\right|_{\theta_L = \bar{\theta}} \leq 0$. As $x_1^{f'}(\theta_L) < 0$, we this is the case only if

$$x_0 q'(x_1^f(\bar{\theta})) + \delta \mathbb{E}\left[\bar{\theta} - \theta\right] q(x_1^f(\bar{\theta})) \geq 0.$$

In this case, $\theta_L$ is given by the solution to

$$x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta \mathbb{E}[\theta_L - \theta | \theta \leq \theta_L] = 0. \tag{32}$$

On the other hand, if

$$x_0 q'(x_1^f(\bar{\theta})) + \delta \mathbb{E}\left[\bar{\theta} - \theta\right] q(x_1^f(\bar{\theta})) < 0,$$

then we get that $\left.\frac{\partial \text{ObjFun}}{\partial \theta_L}\right|_{\theta_L = \bar{\theta}} > 0$, so $\theta_L = \bar{\theta} = 1$.

The next step is to consider the financing constraint. If the constraint is slack, then $\delta = 1$. Let $\theta_L^{\text{uc}}$ the solution when $\delta = 1$. The solution is interior only if

$$x_0 q'(x_1^f(\bar{\theta})) + \left(\bar{\theta} - \mathbb{E}[\theta]\right) q(x_1^f(\bar{\theta})) \geq 0.$$

Noting that when $\bar{\theta} = 1$ we have $x_1^f(\bar{\theta}) = x_0$, this is equivalent to saying that

$$\left.\frac{\partial W_1(x_1, \mathbb{E}[\theta])}{x_1}\right|_{x_1 = x_0} \geq 0,$$

which means that $x_1^*(\mathbb{E}[\theta]) \geq x_0$. If this is the case, the solution from equation (32) becomes

$$x_0 \frac{q'(x_1^f(\theta_L^{\text{uc}}))}{q(x_1^f(\theta_L^{\text{uc}}))} + \mathbb{E}[\theta_L^{\text{uc}} - \theta | \theta \leq \theta_L^{\text{uc}}] = 0$$

On the other hand, if the constraint is binding. Then, $\theta_L = \theta_L^{\text{bc}}$, where $\theta_L^{\text{bc}}$ satysfies

$$x_0 \mathbb{E}\left[q\big(x_1^f(\theta_L^{\text{bc}})\big)\,\mathbf{1}_{\{\theta \le \theta_L^{\text{bc}}\}} + q\big(x_1^f(\theta)\big)\,\mathbf{1}_{\{\theta > \theta_L^{\text{bc}}\}}\right] = I,$$

and from equation (32) we get that

$$\delta = -\frac{x_0}{\mathbb{E}[\theta_L^{\text{bc}} - \theta | \theta \le \theta_L^{\text{bc}}]} \frac{q'(x_1^f(\theta_L^{\text{bc}}))}{q(x_1^f(\theta_L^{\text{bc}}))}.$$

Finally, notting that for any $\theta_L > \theta_L^{\text{bc}}$, we have

$$x_0 \mathbb{E}\left[q\big(x_1^f(\theta_L)\big)\,\mathbf{1}_{\{\theta \le \theta_L\}} + q\big(x_1^f(\theta)\big)\,\mathbf{1}_{\{\theta > \theta_L\}}\right] > I,$$

we conclude that the solution to the optimization problem is $\max\{\theta_L^{\text{bc}}, \theta_L^{\text{uc}}\}$.

□

**Proof of Proposition 4**

*Proof.* Our verification analysis follows the approach in Amador et al. (2006) and Amador and Bagwell (2013). In particular, we find a Lagrange multiplier such that the debt limit and the multiplier are a saddle-point of the Lagrangian. The optimality of the policy then follows from the sufficient conditions in Theorem 2 in (Luenberger, 1969, p. 221). First, we will consider a Lagrange relaxation of the optimization problem in (31). Letting $\lambda$ be the Lagrange multiplier of the financial constraint, we can consider the problem

$$\begin{aligned}
&\max_{x_1(\theta)} \; V_1(x_1(\bar\theta), x_0, \bar\theta) + \mathbb{E}\left[(1+\lambda)x_0 q(x_1(\theta)) - \frac{G(\theta)}{g(\theta)} V_2(x_1(\tilde\theta))\right] - \lambda I \\
&\text{subject to} \\
&V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar\theta), x_0, \bar\theta) - \int_\theta^{\bar\theta} V_2(x_1(\tilde\theta)) d\tilde\theta \quad \forall \theta \in \Theta \\
&x_1(\theta) \ge x_0 \\
&x_1(\theta) \text{ is non-increasing.}
\end{aligned} \tag{33}$$

Dividing by $1/(1+\lambda)$ and defining $\delta \equiv 1/(1+\lambda)$, we consider the problem

$$\begin{aligned}
&\max_{x_1(\theta)} \; \delta V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E}\left[x_0 q(x_1(\theta)) - \delta \frac{G(\theta)}{g(\theta)} V_2(x_1(\tilde{\theta}))\right] \\
&\text{subject to} \\
&V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta})) d\tilde{\theta} \quad \forall \theta \in \Theta \\
&x_1(\theta) \geq x_0 \\
&x_1(\theta) \text{ is non-increasing.}
\end{aligned} \tag{34}$$

If we let $\Lambda(\theta)$ be the cumulative Lagrange multiplier for the constraint

$$V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta})) d\tilde{\theta} \quad \forall \theta \in \Theta,$$

then, the Lagrangian for our problem is

$$\begin{aligned}
\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) \left(\delta + \Lambda(\underline{\theta}) - \Lambda(\bar{\theta})\right) + \int_{\underline{\theta}}^{\bar{\theta}} \left[x_0 q\left(x_1(\theta)\right) g(\theta) - \delta G(\theta) V_2(x_1(\theta))\right] d\theta \\
+ \int_{\underline{\theta}}^{\bar{\theta}} \left(V_1(x_1(\theta), x_0, \theta) + \int_{\theta}^{\bar{\theta}} V_2\left(x_1(\tilde{\theta})\right) d\tilde{\theta}\right) d\Lambda(\theta)
\end{aligned}$$

If we change the order of integration, we obtain that

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} V_2\left(x_1(\tilde{\theta})\right) d\tilde{\theta} d\Lambda(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \Lambda(\theta) V_2(x_1(\theta)) d\theta.$$

Substituting in $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$ we get the following expression for the Lagrangean.

$$\begin{aligned}
\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) \left(\delta + \Lambda(\underline{\theta}) - \Lambda(\bar{\theta})\right) + \int_{\underline{\theta}}^{\bar{\theta}} \left[x_0 q\left(x_1(\theta)\right) g(\theta) + \left(\Lambda(\theta) - \delta G(\theta)\right) V_2(x_1(\theta))\right] d\theta \\
+ \int_{\underline{\theta}}^{\bar{\theta}} V_1(x_1(\theta), x_0, \theta) d\Lambda(\theta)
\end{aligned} \tag{35}$$

Next, we consider the following optimization problem

$$\max_{x_1(\theta)} \ \mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$$
$$x_1(\theta) \geq x_0$$
$$x_1(\theta) \text{ is non-increasing.}$$

Denoting the partial derivative with respect to $x_1$ by $V_1'(x_1, x_0, \theta) \equiv \frac{\partial V_1(x_1, x_0, \theta)}{\partial x_1}$, we get that the directional derivative of $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$ in the direction $\mathbf{h}$ is:

$$\begin{aligned}
&\nabla \mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}; \mathbf{h}) = V_1'(x_1(\bar{\theta}), x_0, \bar{\theta}) \left(\delta + \Lambda(\underline{\theta}) - \Lambda(\bar{\theta})\right) h(\bar{\theta}) \\
&\quad + \int_{\underline{\theta}}^{\bar{\theta}} \left[x_0 q'\left(x_1(\theta)\right) g(\theta) - (\Lambda(\theta) - \delta G(\theta)) q(x_1(\theta))\right] h(\theta) d\theta + \int_{\underline{\theta}}^{\bar{\theta}} V_1'(x_1(\theta), x_0, \theta) h(\theta) d\Lambda(\theta)
\end{aligned} \tag{36}$$

Evaluating the gradient in equation (36) at

$$x_1^l(\theta) \equiv \begin{cases} x_1^f(\theta) & \text{if } \theta \geq \theta_L \\ x_1^f(\theta_L) & \text{if } \theta < \theta_L. \end{cases}$$

we get

$$\begin{aligned}
&\nabla \mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = \int_{\theta_L}^{\bar{\theta}} \left[x_0 q'(x_1^f(\theta)) g(\theta) - (\Lambda(\theta) - \delta G(\theta)) q(x_1^f(\theta))\right] h(\theta) d\theta \\
&+ \int_{\underline{\theta}}^{\theta_L} \left[x_0 q'(x_1^f(\theta_L)) g(\theta) - (\Lambda(\theta) - \delta G(\theta)) q(x_1^f(\theta_L))\right] h(\theta) d\theta + \int_{\underline{\theta}}^{\theta_L} V_1'(x_1^f(\theta_L), x_0, \theta) h(\theta) d\Lambda(\theta)
\end{aligned}$$

Letting

$$\Lambda(\theta) = \begin{cases} \delta & \text{if } \theta = \bar{\theta} \\ \delta G(\theta) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} g(\theta) & \text{if } \theta \in [\theta_L, \bar{\theta}) \\ 0 & \text{if } \theta \in [\underline{\theta}, \theta_L) \end{cases} \tag{37}$$

and noticing that $V_1'(x_1^f(\theta_L), x_0, \theta) = (\theta_L - \theta) q(x_1^f(\theta_L))$, we get

$$\nabla \mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = \int_{\underline{\theta}}^{\theta_L} \left[x_0 q'(x_1^f(\theta_L)) g(\theta) + \delta G(\theta) q(x_1^f(\theta_L))\right] h(\theta) d\theta$$

We want to rewrite the directional derivative $\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h})$ in terms of the increments of $\mathbf{h}(\theta)$, so we can incorporate the monotonicity constraint directly the first order conditions. With this objective in mind, we let

$$\begin{aligned}\Phi(\theta) &\equiv \int_{\underline{\theta}}^{\theta} \left[x_0 q'(x_1^f(\theta_L))g(\tilde{\theta}) + \delta G(\tilde{\theta})q(x_1^f(\theta_L))\right] d\tilde{\theta} \\ &= x_0 q'(x_1^f(\theta_L))G(\theta) + \delta q(x_1^f(\theta_L)) \int_{\underline{\theta}}^{\theta} G(\tilde{\theta})d\tilde{\theta}\end{aligned}$$

and use integration by parts to rewrite the directional derivative as

$$\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = \Phi(\theta_L)h(\theta_L) - \int_{\underline{\theta}}^{\theta_L} \Phi(\theta)dh(\theta).$$

Evaluating $\Phi(\theta)$ at $\theta_L$ we get

$$\Phi(\theta_L) = x_0 q'(x_1^f(\theta_L))G(\theta_L) + \delta q(x_1^f(\theta_L)) \int_{\underline{\theta}}^{\theta} G(\tilde{\theta})d\tilde{\theta} = G(\theta_L)q(x_1^f(\theta_L)) \left[x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta\mathbb{E}[\theta_L - \theta | \theta \leq \theta_L]\right]$$

This expression is equal to zero when $\theta_L$ and $\delta$ are given by Proposition 3. Hence, we can write the directional derivative as

$$\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{h}) = -\int_{\underline{\theta}}^{\theta_L} \Phi(\theta)dh(\theta).$$

If $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$ is concave (we will return to verify this latter), and we let $P$ be set of all non-increasing functions on $\Theta$, then Lemma 1 in (Luenberger, 1969, p. 227) provides the following necessary and sufficient condition for $\mathbf{x}_1^l$ to maximize $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$:

$$\begin{aligned}&\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \tilde{\mathbf{x}}_1) \leq 0 \quad \forall\ \tilde{\mathbf{x}}_1 \in P \\ &\nabla\mathcal{L}(\mathbf{x}_1^l, \mathbf{\Lambda}; \mathbf{x}_1^l) = 0,\end{aligned}$$

As $d\tilde{x}_1(\theta) \leq 0$, we need that $\Phi(\theta) \leq 0$, which is equivalent to requiring that

$$x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta \int_{\underline{\theta}}^{\theta} \frac{G(\tilde{\theta})}{G(\theta)} d\tilde{\theta} \leq 0,\ \forall\theta \in [\underline{\theta}, \theta_L). \tag{38}$$

Using integration by parts, we get

$$\int_{\underline{\theta}}^{\theta} \frac{G(\tilde{\theta})}{G(\theta)} d\tilde{\theta} = \theta - E[\tilde{\theta}|\tilde{\theta} \leq \theta]$$

so can write equation (38) as

$$0 \geq \frac{x_0 q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta E[\theta - \tilde{\theta}|\tilde{\theta} \leq \theta], \quad \forall \theta \in [\underline{\theta}, \theta_L).$$

Substituting $\frac{x_0 q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))}$ from Proposition 3, we get

$$1 \geq \frac{\mathbb{E}[\tilde{\theta}|\tilde{\theta} \leq \theta_L] - \mathbb{E}[\tilde{\theta}|\tilde{\theta} \leq \theta]}{\theta_L - \theta}, \quad \forall \theta \in [\underline{\theta}, \theta_L).$$

As $G(x)$ log-concave (Assumption 1), the function $\theta - \mathbb{E}[\tilde{\theta}|\tilde{\theta} \leq \theta]$ is increasing in $\theta$ (Bagnoli and Bergstrom, 2005) which means that the previous inequality is satisfied for all $\theta < \theta_L$.

It remains to verify the concavity of the $\mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda})$. Substituting the $\mathbf{\Lambda}$ from equation (37) in the Lagrangian in (35), we get

$$\begin{aligned} \mathcal{L}(\mathbf{x}_1, \mathbf{\Lambda}) = & \int_{\underline{\theta}}^{\theta_L} \left[ x_0 q\left(x_1(\theta)\right) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} V_2(x_1(\theta)) \right] g(\theta) d\theta \\ & + \int_{\theta_L}^{\bar{\theta}} \left[ x_0 q\left(x_1(\theta)\right) - \delta \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta)) \right] g(\theta) d\theta + \int_{\underline{\theta}}^{\bar{\theta}} V_1(x_1(\theta), x_0, \theta) d\Lambda(\theta) \end{aligned} \quad (39)$$

Assumption 3 implies that $q(x_1)$ is concave, so the functions $x_0 q\left(x_1\right) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} V_2(x_1)$, $x_0 q\left(x_1\right) - \rho\delta \frac{G(\theta)}{g(\theta)} V_2(x_1)$ and $V_1(x_1, x_0, \theta)$ are concave in $x_1$ for all $\theta \in \Theta$. Hence, for the Lagrangian to be concave, it suffices that $\Lambda(\theta)$ is non-decreasing. This is the case if:

1. On $(\theta_L, \bar{\theta}]$ the function

$$\delta G(\theta) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} g(\theta)$$

is increasing.

2. At $\bar{\theta}$, $\Lambda(\theta_L) \geq \Lambda(\theta_L -)$, which requires that

$$\delta G(\theta_L) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} g(\theta) \geq 0$$

For the second condition, notice that, after substituting the first order condition for $\theta_L$ in Proposition 3, we can write the condition for $\Lambda(\theta_L) \geq \Lambda(\theta_L-)$ as

$$1 - E[\theta_L - \theta | \theta \leq \theta_L] \frac{g(\theta_L)}{G(\theta_L)} = \frac{\partial}{\partial \theta_L} \left[ \frac{\int_{\underline{\theta}}^{\theta_L} G(\theta) d\theta}{G(\theta_L)} \right] \geq 0,$$

which holds whenever $G(\theta)$ is log-concave. □

### Proof of Proposition 5

We consider the optimization problem (26). First, notice because $x_0 q(x_0)$ is concave (from Assumption 3) and $\max_{x_0} x_0 q(x_0) \geq I$ (from Assumption 2), know that that there are two roots $x_0^{\min} < x_0^{\max}$ to the equation $x_0 q(x_0) = I$ such that $x_0 q(x_0) \geq I$ only if $x_0 \in [x_0^{\min}, x_0^{\max}]$. Hence, we can write problem (26) as

$$\max_{x_0 \in [x_0^{\min}, x_0^{\max}]} W_0(x_0) \tag{40}$$

where – given the hypothesis in Proposition 4 – we can write $W_0(x_0)$ as

$$\begin{aligned} W_0(x_0) = \max_{\theta_L \in \Theta} \; & \mathbb{E}\left[ W_1\left( x_1^l(\theta |, x_0, \theta_L), \theta \right) \right] \\ & \text{subject to} \\ & x_0 \mathbb{E}\left[ q\left( \min\{x_1^f(\theta), x_1^f(\theta_L)\} \right) \right] \geq I \end{aligned} \tag{41}$$

Let $W_1'(x_1, \theta) \equiv \frac{\partial W_1(x_1,\theta)}{\partial x_1}$ and let $x_1^L = x_1^f(\theta_L, x_0)$ be the debt limit in Proposition 3. Let $\theta \vee \theta_L \equiv \max\{\theta, \theta_L\}$; by the envelope theorem, we have that for any $x_0 \in [x_0^{\min}, x_0^{\max}]$ the derivative $W_0^{n'}(x_0)$ is

$$\begin{aligned} W_0^{n'}(x_0) = \mathbb{E}\left[ w_1'(x_1^f(\theta \vee \theta_L, x_0), \theta) \frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0} \right] \\ + \lambda \mathbb{E}\left[ q(x_1^f(\theta \vee \theta_L, x_0)) + x_0 q'(x_1^f(\theta \vee \theta_L, x_0)) \frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0} \right] \end{aligned}$$

where $\lambda = \delta^{-1} - 1$ is the multiplier of the financing constraint $x_0 \mathbb{E}\left[ q\left( \min\{x_1^f(\theta), x_1^f(\theta_L)\} \right) \right] \geq I$, and

$$\frac{\partial x_1^f(\theta, x_0)}{\partial x_0} = \begin{cases} \frac{q'(x_1^f(\theta,x_0))}{V_1''(x_1^f(\theta,x_0),x_0,\theta)} & \text{if } \theta < \bar{\theta} = 1 \\ 1 & \text{if } \theta = \bar{\theta} = 1. \end{cases}$$

We have the following result

**Lemma 4.** *Let $x_0^*$ be a solution to problem (40) and $\theta_L^*$ the associated solution to problem (41). If the financing constraint $x_0^* \mathbb{E}\left[q\left(x_1^l(\theta|x_0^*, \theta_L^*)\right)\right] \geq I$ is slack, then $x_0 \geq x_1^*(\mathbb{E}[\theta])$.*

*Proof.* We proceed by contradiction. Suppose that $x_0 < x_1^*(\mathbb{E}[\theta])$. As, $x_1^L = x_1^f(\theta_L, x_0) > x_0$, it must be the case that $x_0 \in (x_0^{\min}, x_0^{\max})$ because $x_0 \in \{x_0^{\min}, x_0^{\max}\}$ together with $x_1^L > x_1^f(\bar{\theta}, x_0) = x_0$ does not satisfy the constraint $x_0 \mathbb{E}\left[q\left(\min\{x_1^f(\theta), x_1^f(\theta_L)\}\right)\right] \geq I$. This means, that $x_0^*$ is in the interior $[x_0^{\min}, x_0^{\max}]$, so it must satisfy the necessary condition $W_0^{n'}(x_0) = 0$. However, if $\lambda = 0$ and $\theta_L < \bar{\theta}$, we have

$$W_0^{n'}(x_0) = \mathbb{E}\left[w_1'(x_1^f(\theta \vee \theta_L, x_0), \theta) \frac{\partial x_1^f(\theta \vee \theta_L, x_0)}{\partial x_0}\right] < 0,$$

yielding a contradiction. □

From here, we can conclude that if the financing constraint is slack. then it must be the case that $x_0 \geq x_1^*(\mathbb{E}[\theta])$. The next lemma shows that in any solution, the constraint must be binding.

**Lemma 5.** *If $x_0^*$ be a solution to problem (40) and $\theta_L^*$ the associated solution to problem (41). Then, the financing constraint is binding, so $x_0^* \mathbb{E}\left[q\left(x_1^l(\theta|x_0^*, \theta_L^*)\right)\right] = I$.*

*Proof.* We proceed by contradiction. Suppose that the constraint is slack. Then, it must be the case that $x_0^* \in (x_0^{\min}, x_0^{\max})$, and by Lemma 4 it must be the case that $x_0^* \geq x_1^*(\mathbb{E}[\theta])$. From Proposition 3, we get that $\theta_L^* = \bar{\theta}$ and $\lambda = 0$. Next, we show that $x_0^*$ cannot achieve a local maximum at $x_0^*$. As $W_1(x_1, \mathbb{E}[\theta])$ is concave in $x_1$, and achieves a maximum at $x_1^*(\mathbb{E}[\theta])$, we have that we have $W_0^{n'}(x_0) < 0$ for any $x_0 > x_1^*(\mathbb{E}[\theta])$, so $x_0^* > x_1^*(\mathbb{E}[\theta])$ cannot be optimal. Hence, if $x_0^* \geq x_1^*(\mathbb{E}[\theta])$ it must be equal to $x_1^*(\mathbb{E}[\theta])$. If $x_0^* = x_1^*(\mathbb{E}[\theta]) \in (x_0^{\min}, x_0^{\max})$, then the constraint must be slack for $x_0 = x_1^*(\mathbb{E}[\theta])$. By continuity, it is also slack for $x_0' = x_1^*(\mathbb{E}[\theta]) - \epsilon$ (for some $\epsilon > 0$), which means that

$$W_0^{n'}(x_0') = \mathbb{E}\left[w_1'(x_1^f(\theta \vee \theta_L, x_0'), \theta) \frac{\partial x_1^f(\theta \vee \theta_L, x_0')}{\partial x_0}\right] < 0.$$

Taking the limit when $\epsilon \to 0$, we get that $W_0^{n'}(x_1^*(\mathbb{E}[\theta]) -) < 0$, which means that $x_1^*(\mathbb{E}[\theta])$ cannot be optimal. We conclude that the constraint cannot be slack at the optimal solution.

□

Next, we provide the necessary condition for the case when $x_0^* > x_0^{\min}$.

**Lemma 6.** *If the financing constraint is binding and $x_0 > x_0^{\min}$, then $x_0$ and $\theta_L$ satisfies the following necessary conditions*

$$x_0 = \frac{(1-\delta)\mathbb{E}\left[q(x_1^f(\theta_L, x_0))\mathbf{1}_{\{\theta\le\theta_L\}} + q(x_1^f(\theta, x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right]}{-\mathbb{E}\left[q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta,x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}}\right]}$$

$$\delta = -\frac{x_0}{\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]}\frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))}$$

$$x_0\mathbb{E}\left[q(x_1^f(\theta_L, x_0))\mathbf{1}_{\{\theta\le\theta_L\}} + q(x_1^f(\theta, x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right] = I.$$

*Proof.* If the solution is interior, it must be the case that $W_0^{n'}(x_0) = 0$. If $x_1^L = x_1^f(\theta_L, x_0) > x_0$, then we can write the necessary conditions as

$$W_0^{n'}(x_0) = \mathbb{E}\left[w_1'(x_1^f(\theta, x_0), \theta)\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}} + w_1'(x_1^f(\theta_L, x_0), \theta)\frac{\partial x_1^f(\theta_L, x_0)}{\partial x_0}\mathbf{1}_{\{\theta\le\theta_L\}}\right]$$
$$+\lambda\mathbb{E}\left[q(x_1^l(\theta, x_0|x_1^L) + x_0 q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}} + x_0 q'(x_1^f(\theta_L, x_0))\frac{\partial x_1^f(\theta_L, x_0)}{\partial x_0}\mathbf{1}_{\{\theta\le\theta_L\}}\right] = 0$$

For $\theta > \theta_L$, we have that

$$w_1'(x_1^f(\theta, x_0), \theta) = x_0 q'(x_1^f(\theta, x_0)) + V_1'(x_1^f(\theta, x_0), x_0, \theta) = x_0 q'(x_1^f(\theta, x_0));$$

while for $\theta < \theta_L$ we have

$$w_1'(x_1^f(\theta, x_0), \theta) = x_0 q'(x_1^f(\theta, x_0)) + V_1'(x_1^f(\theta, x_0), x_0, \theta) = x_0 q'(x_1^f(\theta, x_0)) + (\theta_L - \theta)q(x_1^f(\theta_L, x_0))$$

Hence, we can write the equation $W_0^{n'}(x_0) = 0$ as

$$(1+\lambda)\mathbb{E}\left[x_0 q'(x_1^f(\theta, x_0))\frac{\partial x_1^f(\theta, x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}} + x_0 q'(x_1^f(\theta_L, x_0))\frac{\partial x_1^f(\theta_L, x_0)}{\partial x_0}\mathbf{1}_{\{\theta<\theta_L\}}\right]$$
$$\mathbb{E}[(\theta_L-\theta)\mathbf{1}_{\{\theta<\theta_L\}}]q(x_1^f(\theta_L, x_0))\frac{\partial x_1^f(\theta_L, x_0)}{\partial x_0} + \lambda\mathbb{E}\left[q(x_1^f(\theta_L, x_0))\mathbf{1}_{\{\theta\le\theta_L\}} + q(x_1^f(\theta, x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right] = 0$$

Noting that $\mathbb{E}[(\theta_L-\theta)\mathbf{1}_{\{\theta<\theta_L\}}]=G(\theta_L)\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]$, and letting $\lambda=\delta^{-1}-1$, we can write

$$\mathbb{E}\left[x_0q'(x_1^f(\theta,x_0))\frac{\partial x_1^f(\theta,x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}}\right]+G(\theta_L)x_0q'(x_1^f(\theta_L,x_0))\frac{\partial x_1^f(\theta_L,x_0)}{\partial x_0}$$
$$\delta G(\theta_L)\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]q(x_1^f(\theta_L,x_0))\frac{\partial x_1^f(\theta_L,x_0)}{\partial x_0}+(1-\delta)\mathbb{E}\left[q(x_1^f(\theta_L,x_0))\mathbf{1}_{\{\theta\le\theta_L\}}+q(x_1^f(\theta,x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right]=0$$

From Proposition 3, $\delta$ is given by

$$\delta=-\frac{x_0}{\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]}\frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))},$$

which means that

$$G(\theta_L)x_0q'(x_1^f(\theta_L,x_0))\frac{\partial x_1^f(\theta_L,x_0)}{\partial x_0}+\delta G(\theta_L)\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]q(x_1^f(\theta_L,x_0))\frac{\partial x_1^f(\theta_L,x_0)}{\partial x_0}=0.$$

From here, we that the first order condition for $x_0$ can be written as

$$x_0=\frac{(1-\delta)\mathbb{E}\left[q(x_1^f(\theta_L,x_0))\mathbf{1}_{\{\theta\le\theta_L\}}+q(x_1^f(\theta,x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right]}{-\mathbb{E}\left[q'(x_1^f(\theta,x_0))\frac{\partial x_1^f(\theta,x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}}\right]}$$

We conclude then that $(x_0,\theta_L,\delta)$ satisfy

$$\begin{aligned}
x_0&=\frac{(1-\delta)\mathbb{E}\left[q(x_1^f(\theta_L,x_0))\mathbf{1}_{\{\theta\le\theta_L\}}+q(x_1^f(\theta,x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right]}{-\mathbb{E}\left[q'(x_1^f(\theta,x_0))\frac{\partial x_1^f(\theta,x_0)}{\partial x_0}\mathbf{1}_{\{\theta>\theta_L\}}\right]}\\
\delta&=-\frac{x_0}{\mathbb{E}[\theta_L-\theta|\theta\le\theta_L]}\frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))}\\
x_0&\mathbb{E}\left[q(x_1^f(\theta_L,x_0))\mathbf{1}_{\{\theta\le\theta_L\}}+q(x_1^f(\theta,x_0))\mathbf{1}_{\{\theta>\theta_L\}}\right]=I.
\end{aligned}$$

□

The final step is to characterize the conditions under which $x_0^*>x_0^{\min}$.

**Lemma 7.** *Let $x_0^*$ be a solution to the problem (40). If $x_0^{\min}\ge x_1^*(\mathbb{E}[\theta])$; then $x_0^*$, while if $x_0^{\min}<x_1^*(\mathbb{E}[\theta])$ then $x_0^*>x_0^{\min}$.*

*Proof.* We consider two cases: $x_0^{\min}\ge x_1^*(\mathbb{E}[\theta])$ and $x_0^{\min}<x_1^*(\mathbb{E}[\theta])$. If $x_0^{\min}\ge x_1^*(\mathbb{E}[\theta])$, then

Proposition 3 implies that for any $x_0 \in [x_0^{\min}, x_0^{\max}]$, we have $\theta_L = \bar{\theta} = 1$, which means that $x_0^* = \arg\max_{x_0 \in [x_0^{\min}, x_0^{\max}]} W_1(x_0, \mathbb{E}[\theta])$. As $W_1(x_0, \mathbb{E}[\theta])$ is a concave function (by Assumption 3) achieving its maximum at $x_1^*(\mathbb{E}[\theta]) \leq x_0^{\min}$, it follows that $x_0^* = x_0^{\min}$. On the other hand, if $x_0^{\min} \geq x_1^*(\mathbb{E}[\theta])$, then $W_1(x_0, \mathbb{E}[\theta])$ is increasing at $x_0 = x_0^{\min}$, which means tha $x_0^* > x_0^{\min}$ □

# C Extensions

## C.1 Short-Term Debt

Motivated by the solution in Section 4, we consider contracts $(b, x_0, x_1^L)$, where $b$ is the amount of short-term debt that matures at $t = 1$, $x_0$ is the initial amount of long-term debt that matures at $t = 2$, and $x_1^L$ is the limit in total debt due at $t = 2$ (which limits the amount of new issuance at $t = 1$). As we did before, it is convenient to consider the problem in terms of the marginal types that are constrained at $t = 1$. As before, we have the critical type $\theta_L$ satisfying $x_1^f(\theta_L) = x_1^L$ – for which the debt limit binds. Because the borrower is financially constrained, the following roll-over constraint must be satisfied at $t = 1$:

$$q(x_1(\theta))(x_1(\theta) - x_0) \geq b.$$

This constraint specifies that the amount rise at $t = 1$ must be sufficient to pay back the maturing short-term debt. It is possible to roll over the debt only if $q(x_1^L)(x_1^L - x_0) \geq b$. Moreover, if $\bar{\theta} = 1$ (Assumption 5) we have that $x_1^f(\bar{\theta}) - x_0 = 0$, which means that there is a marginal type $\theta_H$ such that

$$q(x_1^f(\theta_H))(x_1^f(\theta_H) - x_0) = b.$$

Hence, we can consider the following generalization of the problem in equation (23)

$$\begin{aligned}
&\max_{\substack{\theta_L, \theta_H \in \Theta \\ \theta_L \leq \theta_H}} \int_{\underline{\theta}}^{\theta_L} W_1(x_1^f(\theta_L), \theta) dG(\theta) + \int_{\theta_L}^{\theta_H} W_1(x_1^f(\theta), \theta) dG(\theta) + \int_{\theta_H}^{\bar{\theta}} W_1(x_1^f(\theta_H), \theta) dG(\theta) \\
&\text{subject to} \\
&x_0 \left[ \left( q(x_1^f(\theta_L)) - q(x_1^f(\theta_H)) \right) G(\theta_L) + \int_{\theta_L}^{\theta_H} \left( q(x_1^f(\theta)) - q(x_1^f(\theta_H)) \right) dG(\theta) \right] \\
&+ x_1^f(\theta_H) q(x_1^f(\theta_H)) \geq I.
\end{aligned} \tag{42}$$

Let $\lambda$ be the multiplier of the investment constraint. We can write

$$\max_{\substack{\theta_L,\theta_H\in\Theta\\ \theta_L\le\theta_H}} \int_{\underline{\theta}}^{\theta_L} \Big(W_1(x_1^f(\theta_L),\theta)+\lambda x_0 q(x_1^f(\theta_L))\Big)\, dG(\theta) + \int_{\theta_L}^{\theta_H} \Big(W_1(x_1^f(\theta),\theta)+\lambda x_0 q(x_1^f(\theta))\Big)\, dG(\theta)$$
$$+\int_{\theta_H}^{\bar{\theta}} \Big(W_1(x_1^f(\theta_H),\theta)+\lambda x_0 q(x_1^f(\theta_H))\Big)\, dG(\theta) + \lambda q(x_1^f(\theta_H))(x_1^f(\theta_H)-x_0)$$

Dividing by $1+\lambda$ and letting $\delta = 1/(1+\lambda)$, we get that the previous problem is equivalent to

$$\max_{\substack{\theta_L,\theta_H\in\Theta\\ \theta_L\le\theta_H}} \int_{\underline{\theta}}^{\theta_L} \Big((1-\delta)x_0 q(x_1^f(\theta_L))+\delta W_1(x_1^f(\theta_L),\theta)\Big)\, dG(\theta) + \int_{\theta_L}^{\theta_H} \Big((1-\delta)x_0 q(x_1^f(\theta))+\delta W_1(x_1^f(\theta),\theta)\Big)\, dG(\theta)$$
$$+\int_{\theta_H}^{\bar{\theta}} \Big((1-\delta)x_0 q(x_1^f(\theta_H))+\delta W_1(x_1^f(\theta_H),\theta)\Big)\, dG(\theta) + (1-\delta) q(x_1^f(\theta_H))(x_1^f(\theta_H)-x_0)$$

If we ignore the constraint $\theta_L \le \theta_H$, the first order condition for $\theta_L$ is the same as in the case without short-term debt. So, we get the condition

$$x_0\frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta\mathbb{E}[\theta_L-\theta|\theta\le\theta_L] = 0$$

Next, we consider the first-order condition for $\theta_H$; which is given by

$$\begin{aligned}\frac{\partial \text{ObjFun}}{\partial\theta_H} &= x_1^{f'}(\theta_H)\left\{\int_{\theta_H}^{\bar{\theta}}\Big(x_0 q'(x_1^f(\theta_H))+\delta(\theta_H-\theta)q(x_1^f(\theta_H))\Big)\, dG(\theta) + (1-\delta)\theta_H q(x_1^f(\theta_H))\right\}\\ &= (1-G(\theta_H))x_1^{f'}(\theta_H)q(x_1^f(\theta_H))\left\{x_0\frac{q'(x_1^f(\theta_H))}{q(x_1^f(\theta_H))} - \delta\mathbb{E}[\theta-\theta_H|\theta\ge\theta_H] + (1-\delta)\frac{\theta_H}{1-G(\theta_H)}\right\} = 0\end{aligned}$$

where we have used the following relationships

$$\begin{aligned}&x_1^f(\theta_H)q'(x_1^f(\theta_H)) + (1-\theta)q(x_1^f(\theta_H)) = (\theta_H-\theta)q(x_1^f(\theta_H)) + x_0 q'(x_1^f(\theta))\\ &(x_1^f(\theta_H)-x_0)q'(x_1^f(\theta_H)) + q(x_1^f(\theta_H)) = \theta_H q(x_1^f(\theta_H))\end{aligned}$$

From here, we get the first-order condition

$$x_0\frac{q'(x_1^f(\theta_H))}{q(x_1^f(\theta_H))} - \delta\mathbb{E}[\theta-\theta_H|\theta\ge\theta_H] + (1-\delta)\frac{\theta_H}{1-G(\theta_H)} = 0$$

Notice that if $\delta = 1$, then

$$x_0 \frac{q'(x_1^f(\theta_H))}{q(x_1^f(\theta_H))} - \delta \mathbb{E}[\theta - \theta_H | \theta \geq \theta_H] < 0,$$

which means that $\frac{\partial \text{ObjFun}}{\partial \theta_H} > 0$ so $\theta_H = \bar{\theta}$. On the other hand, if the financing constraint is binding (that is, $\delta < 1$) then $(\theta_L, \theta_H, \delta)$ satisfy

$$\begin{aligned}
& x_0 \frac{q'(x_1^f(\theta_H))}{q(x_1^f(\theta_H))} - \delta \mathbb{E}[\theta - \theta_H | \theta \geq \theta_H] + (1-\delta) \frac{\theta_H}{1 - G(\theta_H)} = 0 \\
& x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta \mathbb{E}[\theta_L - \theta | \theta \leq \theta_L] = 0 \\
& x_0 \left[ G(\theta_L) q(x_1^f(\theta_L)) + \int_{\theta_L}^{\theta_H} q(x_1^f(\theta)) dG(\theta) - G(\theta_H) q(x_1^f(\theta_H)) \right] + x_1^f(\theta_H) q(x_1^f(\theta_H)) = I
\end{aligned}$$

The next step is to consider the first-order condition for $x_0$. Let $W_0(x_0)$ be the value function of the optimization problem in equation (42). From here, we get

$$\begin{aligned}
W_0'(x_0) = {} & x_0 \int_{\theta_L}^{\theta_H} q'(x_1^f(\theta)) \frac{\partial x_1^f(\theta)}{\partial x_0} dG(\theta) + \frac{\partial x_1^f(\theta_L)}{\partial x_0} \int_{\underline{\theta}}^{\theta_L} \left( x_0 q'(x_1^f(\theta_L)) + \delta(\theta_L - \theta) q(x_1^f(\theta_L)) \right) dG(\theta) \\
& + \frac{\partial x_1^f(\theta_H)}{\partial x_0} \left\{ \int_{\theta_H}^{\bar{\theta}} \left( x_0 q'(x_1^f(\theta_H)) + \delta(\theta_H - \theta) q(x_1^f(\theta_H)) \right) dG(\theta) + (1-\delta) \theta_H q(x_1^f(\theta_H)) \right\} \\
& + (1-\delta) \left[ \int_{\underline{\theta}}^{\theta_L} q(x_1^f(\theta_L)) dG(\theta) + \int_{\theta_L}^{\theta_H} q(x_1^f(\theta)) dG(\theta) + \int_{\theta_H}^{\bar{\theta}} q(x_1^f(\theta_H)) dG(\theta) - q(x_1^f(\theta_H)) \right] \\
= {} & x_0 \int_{\theta_L}^{\theta_H} \underbrace{q'(x_1^f(\theta)) \frac{\partial x_1^f(\theta)}{\partial x_0}}_{<0} dG(\theta) \\
& + (1-\delta) \left[ \underbrace{\left( q(x_1^f(\theta_L)) - q(x_1^f(\theta_H)) \right)}_{<0} G(\theta_L) + \int_{\theta_L}^{\theta_H} \underbrace{\left( q(x_1^f(\theta)) - q(x_1^f(\theta_H)) \right)}_{<0} dG(\theta) \right] < 0
\end{aligned}$$

From here, we get that the solution is $x_0 = 0$, so $x_1^f(\theta) = x_1^*(\theta)$. Substituting back into the first order conditions for $\theta_L$ and $\theta_H$ we get that $\theta_L = \underline{\theta}$ and

$$\begin{aligned}
x_1^*(\theta_H)q(x_1^*(\theta_H)) &= I \\
\delta &= \frac{\theta_H}{\theta_H + \mathbb{E}[\theta - \theta_H|\theta \geq \theta_H](1 - G(\theta_H))}
\end{aligned}$$

**Proof of Proposition 6**

*Proof.* In the linear case, $(x_0, \theta_L)$ solve

$$\begin{aligned}
I &= \frac{x_0(Y - x_0)}{c}\mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta \leq \theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta > \theta_L\}}\right] \\
\frac{x_0}{Y - x_0} &= \frac{(1-\delta)\mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta \leq \theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta > \theta_L\}}\right]}{\mathbb{E}\left[\frac{1}{2-\theta}\mathbf{1}_{\{\theta > \theta_L\}}\right]} \\
\delta &= \frac{2-\theta_L}{\mathbb{E}[\theta_L - \theta|\theta \leq \theta_L]}\frac{x_0}{Y - x_0}.
\end{aligned}$$

Let

$$\begin{aligned}
F(\theta_L) &\equiv \mathbb{E}\left[\frac{1}{2-\theta_L}\mathbf{1}_{\{\theta \leq \theta_L\}} + \frac{1}{2-\theta}\mathbf{1}_{\{\theta > \theta_L\}}\right] \\
\rho(\theta_L) &\equiv \mathbb{E}[\theta_L - \theta|\theta \leq \theta_L].
\end{aligned}$$

Notice that as $G(\theta)$ is log-concave, the function $\rho(\theta)$ is increasing. We can write the system as

$$\begin{aligned}
I &= \frac{x_0(Y - x_0)}{c}F(\theta_L) \\
\frac{x_0}{Y - x_0} &= \frac{(1-\delta)F(\theta_L)}{F(\theta_L) - \frac{G(\theta_L)}{2-\theta_L}} \\
\delta &= \frac{2-\theta_L}{\rho(\theta_L)}\frac{x_0}{Y - x_0}.
\end{aligned}$$

From the first equation, we get

$$x_0^2 - x_0 Y + \frac{cI}{F(\theta_L)} = 0$$

The lowest root is given by

$$x_0 = \frac{Y - \sqrt{Y^2 - \frac{4cI}{F(\theta_L)}}}{2}.$$

Letting

$$\xi = \frac{cI}{Y^2},$$

we can write

$$\frac{x_0}{Y} = \frac{1 - \sqrt{1 - \frac{\xi}{F(\theta_L)}}}{2}.$$

The condition

$$\mathbb{E}[\theta] < \frac{2\sqrt{Y^2 - 4cI}}{Y + \sqrt{Y^2 - 4cI}} \iff \mathbb{E}[\theta] < \frac{2\sqrt{1 - 4\xi}}{1 + \sqrt{1 - 4\xi}}$$

Substituting $\delta$ in the second equation, we get

$$\frac{x_0}{Y - x_0} = \left(1 - \frac{2 - \theta_L}{\rho(\theta_L)} \frac{x_0}{Y - x_0}\right) \frac{F(\theta_L)}{F(\theta_L) - \frac{G(\theta_L)}{2 - \theta_L}}$$

$$\implies$$

$$\frac{x_0}{Y} = \frac{F(\theta_L)}{2F(\theta_L) + \frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}}$$

Combining both expressions for $x_0/Y$, we get

$$\frac{1 - \sqrt{1 - \frac{4\xi}{F(\theta_L)}}}{2} = \frac{F(\theta_L)}{2F(\theta_L) + \frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}}$$

$$\implies$$

$$\frac{1}{1 + \frac{2F(\theta_L)}{\frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}}} = \sqrt{1 - \frac{4\xi}{F(\theta_L)}}$$

Letting

$$\omega(\theta_L) \equiv \frac{2F(\theta_L)}{\frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}} = \frac{2F(\theta_L)\rho(\theta_L)(2 - \theta_L)}{(2 - \theta_L)^2 - G(\theta_L)\rho(\theta_L)},$$

we can write the equation for $\theta_L$ as

$$f(\theta_L, \xi) \equiv \frac{1}{1 + \omega(\theta_L)} - \sqrt{1 - \frac{4\xi}{F(\theta_L)}} = 0$$

At $\theta_L = \underline{\theta}$, we get $\rho(\underline{\theta}) = 0$ so $\omega(\underline{\theta}) = 0$ and $F(\underline{\theta}) = \mathbb{E}[1/(2-\theta)]$ , which means that

$$f(\underline{\theta}, \xi) = \frac{1}{1+\omega(\underline{\theta})} - \sqrt{1 - \frac{4\xi}{F(\underline{\theta})}} = 1 - \sqrt{1 - \frac{4\xi}{F(\underline{\theta})}} > 0$$

On the other hand, if we take $\theta_L = \bar{\theta} = 1$, we get $\rho(\bar{\theta}) = 1 - \mathbb{E}[\theta]$ and $F(\bar{\theta}) = 1/(2-\bar{\theta}) = 1$ so

$$\omega(\bar{\theta}) = \frac{2(1-\mathbb{E}[\theta])}{\mathbb{E}[\theta]}$$

which yields

$$f(\bar{\theta}, \xi) = \frac{1}{1+\omega(\bar{\theta})} - \sqrt{1 - \frac{4\xi}{F(\bar{\theta})}} = \frac{\mathbb{E}[\theta]}{2-\mathbb{E}[\theta]} - \sqrt{1-4\xi},$$

which is negative if

$$\mathbb{E}[\theta] < \frac{2\sqrt{1-4\xi}}{1+\sqrt{1-4\xi}}$$

which corresponds to the condition $x_1^*(\mathbb{E}[\theta]) > x_0^{\min}$. Hence, we get that $f(\underline{\theta}, \xi) > 0$ and $f(\bar{\theta}, \xi) < 0$. Differentiating with respect to $\xi$, we get

$$\frac{\partial f(\theta, \xi)}{\partial \xi} = \frac{1}{2F(\theta)} \left(1 - \frac{4\xi}{F(\theta_L)}\right)^{-1/2} > 0$$

It follows from Theorem 1 in Milgrom and Roberts (1994) that $\theta_L = \min\{\theta \in \Theta : f(\theta, \xi) \leq 0\}$ is non-decreasing in $\xi$. The next step is to look at the comparative statics for $x_0$. As

$$\frac{x_0}{Y} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\xi}{F(\theta_L)}},$$

the sign of the comparative statics for $x_0$ depends on the sign of the comparative statics of $\sqrt{1 - \frac{4\xi}{F(\theta_L)}}$. Moreover, when $f(\theta_L, \xi) = 0$, we have

$$\sqrt{1 - \frac{4\xi}{F(\theta_L)}} = \frac{1}{1+\omega(\theta_L)},$$

so the comparative statics for $x_0$ depends on the sign for $\omega'(\theta_L)$. The derivative of $\omega(\theta_L)$ is

$$\begin{aligned}\omega'(\theta_L) &= \frac{2F'(\theta_L)}{\frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}} - \frac{2F(\theta_L)}{\left[\frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}\right]^2}\left[-\frac{1}{\rho(\theta_L)} - \frac{(2-\theta_L)\rho'(\theta_L)}{\rho(\theta_L)^2} - \frac{g(\theta_L)}{2-\theta_L} - \frac{G(\theta_L)}{(2-\theta_L)^2}\right] \\ &= \frac{2F'(\theta_L)}{\frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}} + \frac{2F(\theta_L)}{\left[\frac{2-\theta_L}{\rho(\theta_L)} - \frac{G(\theta_L)}{2-\theta_L}\right]^2}\left[\frac{1}{\rho(\theta_L)} + \frac{(2-\theta_L)\rho'(\theta_L)}{\rho(\theta_L)^2} + \frac{g(\theta_L)}{2-\theta_L} + \frac{G(\theta_L)}{(2-\theta_L)^2}\right]\end{aligned}$$

The derivative $F'(\theta)$ is

$$F'(\theta_L) = \frac{G(\theta_L)}{(2-\theta_L)^2} > 0,$$

which means that $\omega'(\theta_L) > 0$, it follows then that

$$\left[\sqrt{1 - \frac{\xi}{F(\theta_L(\xi))}}\right]' = \left[\frac{1}{1+\omega(\theta_L(\xi))}\right]' = -\frac{\omega'(\theta_L)}{(1+\omega(\theta_L))^2}\theta_L'(\xi) \leq 0,$$

so

$$\frac{\partial}{\partial \xi}\frac{x_0(\xi)}{Y} \geq 0$$

□

## C.2 Mandatory Prepayment

### Proof of Proposition 8

*Proof.* Using the envelope theorem, the expected transfer at $t = 1$ is

$$m_1(\theta) = V_1(x_1^+(\theta), x_0, \theta) - \bar{V}_1 + \int_\theta^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta},$$

where $\bar{V}_1 = V_1(x_1^+(\bar{\theta}), x_0, \bar{\theta}) - m(\bar{\theta})$. Substituting in the budget constraint and using integration by parts, we get

$$\mathbb{E}\left[x_0 q(x_1^+(\theta)) + m_1(\theta)\right] = \mathbb{E}\left[W_1(x_1^+(\theta), \theta) + V_2(x_1^+(\theta))\frac{G(\theta)}{g(\theta)}\right] - \bar{V}_1$$

Moreover, noticing that

$$V_1(x_1^+(\theta), x_0, \theta) = (x_1^+(\theta) - x_0)q(x_1^+(\theta)) + \theta V_2(x_1^+(\theta)),$$

we can write the constraint $(x_1^+(\theta) - x_0)q(x_1^+(\theta)) \geq m_1(\theta)$ as

$$\bar{V}_1 - \theta V_2(x_1^+(\theta)) - \int_\theta^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta} \geq 0$$

Notice that

$$m_1'(\theta) = \underbrace{\frac{V_1(x_1^+(\theta), x_0, \theta)}{\partial x_1}}_{=(x_1^+(\theta)-x_0)q'(x_1^+(\theta))+(1-\theta)q(x_1^+(\theta))} \qquad x_1^{+'}(\theta) \leq 0,$$

hence, the constraint $m_1(\theta) \geq 0$ only needs to be enforced at $\theta = \bar{\theta}$, were $m_1(\bar{\theta}) = V_1(x_1^+(\bar{\theta}), x_0, \bar{\theta}) - \bar{V}_1$. If $x_1^+(\theta)$ is non-increasing, then $\int_\theta^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta} + \theta V_2(z_1(\theta))$ is increasing, so we only need to enforece the constraint $(x_1^+(\theta) - x_0)q(x_1^+(\theta)) \geq m_1(\theta)$ at $\theta = \bar{\theta}$.

It follows that we can write the mechanism design problem as

$$\begin{aligned}
&\max_{x_0, x_1^+(\theta), m_1(\theta)} \mathbb{E}\left[W_1(x_1^+(\theta), \theta)\right] \\
&\text{subject to} \\
&\mathbb{E}\left[W_1(x_1^+(\theta), \theta) + V_2(x_1^+(\theta))\frac{G(\theta)}{g(\theta)}\right] - \bar{V}_1 \geq I \\
&\overline{V}_1 \in [V_1(x_1^+(\bar{\theta}), x_0, \bar{\theta}), \bar{\theta}V_2(x_1^+(\bar{\theta}))] \\
&x_1^+(\theta) \text{ is non-increasing}
\end{aligned}$$

The previous problem is feasible only if $V_1(x_1^+(\bar{\theta}), x_0, \bar{\theta}) \geq \bar{\theta}V_2(x_1^+(\bar{\theta}))$ which immeidately yields $x_0 = x_1^+(\bar{\theta})$ and $\overline{V}_1 = \bar{\theta}V_2(x_1^+(\bar{\theta}))$. Hence, we get

$$\begin{aligned}
&\max_{x_0, x_1^+(\theta), m_1(\theta)} \mathbb{E}\left[W_1(x_1^+(\theta), \theta)\right] \\
&\text{subject to} \\
&\mathbb{E}\left[W_1(x_1^+(\theta), \theta) + V_2(x_1^+(\theta))\frac{G(\theta)}{g(\theta)}\right] - \bar{\theta}V_2(x_1^+(\bar{\theta})) \geq I \\
&x_1^+(\theta) \text{ is non-increasing}
\end{aligned}$$

As we did before, it is convenient to define the shadow discount factor $\delta \equiv 1/(1+\lambda)$, where $\lambda$ is the multiplier of the budget constraint. Then, we can write the optimization problem for given $\delta$

as

$$\max_{x_1^+(\theta), m_1(\theta)} \mathbb{E}\left[W_1(x_1^+(\theta),\theta) + (1-\delta)\frac{G(\theta)}{g(\theta)}V_2(x_1^+(\theta))\right] - (1-\delta)\bar{\theta}V_2(x_1^+(\bar{\theta}))$$

subject to

$x_1^+(\theta)$ is non-increasing

There is one constraint that we have not considered yet. The maximum possible mandatory prepayment is $b(\theta) = x_0$. This means that $m_1(\theta) \leq (1 - q(x_1^+(\theta)))x_0$ which requires that

$$V_1(x_1^+(\theta), x_0, \theta) - \bar{V}_1 + \int_{\theta}^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta} \leq (1 - q(x_1^+(\theta)))x_0$$

Substituting

$$V_1(x_1^+(\theta), x_0, \theta) = W_1(x_1^+(\theta), \theta) - x_0 q(x_1^+(\theta))$$

we get

$$W_1(x_1^+(\theta), \theta) - \bar{V}_1 + \int_{\theta}^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta} \leq x_0$$

Let

$$H(\theta) \equiv W_1(x_1^+(\theta), \theta) - \bar{V}_1 + \int_{\theta}^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta}$$

So,

$$H'(\theta) = W_1'(x_1^+(\theta), \theta)x_1^{+'}(\theta)$$

Evaluated at

$$x_1^+(\theta) \equiv \arg\max_{x_1} \left\{W_1(x_1, \theta) + (1-\delta)\frac{G(\theta)}{g(\theta)}V_2(x_1)\right\}$$

we get

$$W_1'(x_1^+(\theta), \theta) = (1-\delta)\frac{G(\theta)}{g(\theta)}q(x_1^+) > 0$$

so for $\theta \in [\underline{\theta}, \theta_H]$ we have $H'(\theta) < 0$. From here, we get that the exclusive financing solution can be implemented if $H(\underline{\theta}) \leq x_0$, which means that

$$W_1(x_1^+(\underline{\theta}), \underline{\theta}) - \theta_H V_2(x_1^+(\theta_H)) + \int_{\underline{\theta}}^{\theta_H} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta} \leq x_1^+(\theta_H)$$

If this condition is satisfied, then the solution to the single-lender problem also solves the problem with mandatory prepayment.

If this condition is not satisfied, then we need to consider the problem

$$\max_{x_1^+(\theta)} \mathbb{E}\left[W_1(x_1^+(\theta),\theta) + (1-\delta)\frac{G(\theta)}{g(\theta)}V_2(x_1^+(\theta))\right] - (1-\delta)\bar{\theta}V_2(x_1^+(\bar{\theta}))$$

subject to

$$W_1(x_1^+(\theta),\theta) - \bar{\theta}V_2(x_1^+(\bar{\theta})) + \int_\theta^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta} \leq x_1^+(\bar{\theta})$$

$x_1^+(\theta)$ is non-increasing

Let $\Gamma(\theta)$ be the cumulative multiplier of the constraint. Then, we can write the Lagrangian as

$$\begin{aligned}\mathcal{L}(\mathbf{x}_1,\mathbf{\Gamma}) = &-(1-\delta)\bar{\theta}V_2(x_1^+(\bar{\theta})) + \int_{\underline{\theta}}^{\bar{\theta}} \left[W_1(x_1^+(\theta),\theta)g(\theta) + (1-\delta)G(\theta)V_2(x_1^+(\theta))\right] d\theta \\ &+ \left(\Gamma(\bar{\theta}) - \Gamma(\underline{\theta})\right)\left(x_1^+(\bar{\theta}) + \bar{\theta}V_2(x_1^+(\bar{\theta}))\right) - \int_{\underline{\theta}}^{\bar{\theta}} \left[W_1(x_1^+(\theta),\theta) + \int_\theta^{\bar{\theta}} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta}\right] d\Gamma(\theta).\end{aligned}$$

Changing the order of integration

$$\int_{\underline{\theta}}^{\bar{\theta}}\int_\theta^{\bar{\theta}} V_2\left(x_1(\tilde{\theta})\right)d\tilde{\theta}d\Gamma(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \Gamma(\theta)V_2(x_1(\theta))d\theta.$$

so we get

$$\begin{aligned}\mathcal{L}(\mathbf{x}_1,\mathbf{\Gamma}) = &-(1-\delta)\bar{\theta}V_2(x_1^+(\bar{\theta})) + \int_{\underline{\theta}}^{\bar{\theta}} \left[W_1(x_1^+(\theta),\theta)g(\theta) + \left((1-\delta)G(\theta) - \Gamma(\theta)\right)V_2(x_1^+(\theta))\right] d\theta \\ &+ \left(\Gamma(\bar{\theta}) - \Gamma(\underline{\theta})\right)\left(x_1^+(\bar{\theta}) + \bar{\theta}V_2(x_1^+(\bar{\theta}))\right) - \int_{\underline{\theta}}^{\bar{\theta}} W_1(x_1^+(\theta),\theta)d\Gamma(\theta)\end{aligned}$$

The directional derivative is

$$\begin{aligned}\nabla\mathcal{L}(\mathbf{x}_1,\mathbf{\Gamma};\mathbf{h}) &= \left[\left(\Gamma(\bar{\theta}) - \Gamma(\underline{\theta}) - (1-\delta)\right)\bar{\theta}V_2'(x_1^+(\bar{\theta})) + \left(\Gamma(\bar{\theta}) - \Gamma(\underline{\theta})\right)\right]h(\bar{\theta}) \\ &+ \int_{\underline{\theta}}^{\bar{\theta}} \left[W_1'(x_1^+(\theta),\theta)g(\theta) + \left((1-\delta)G(\theta) - \Gamma(\theta)\right)V_2'(x_1^+(\theta))\right] h(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} W_1'(x_1^+(\theta),\theta)h(\theta)d\Gamma(\theta)\end{aligned}$$

Substituting $V_2'(x) = -q(x)$ we get

$$\nabla\mathcal{L}(\mathbf{x}_1, \mathbf{\Gamma}; \mathbf{h}) = \left[-\left(\Gamma(\bar{\theta}) - \Gamma(\underline{\theta}) - (1-\delta)\right)\bar{\theta}q(x_1^+(\bar{\theta})) + \left(\Gamma(\bar{\theta}) - \Gamma(\underline{\theta})\right)\right]h(\bar{\theta})$$
$$+ \int_{\underline{\theta}}^{\bar{\theta}} \left[W_1'(x_1^+(\theta), \theta)g(\theta) - \left((1-\delta)G(\theta) - \Gamma(\theta)\right)q(x_1^+(\theta))\right]h(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} W_1'(x_1^+(\theta), \theta)h(\theta)d\Gamma(\theta)$$

When the constraint is binding in an interval $(\theta', \theta'')$ then it must be the case that

$$W_1'(x_1^+(\theta), \theta){x_1^+}'(\theta) = 0,$$

so either $x_1^+(\theta) = x_1^*(\theta)$ or ${x_1^+}'(\theta) = 0$. Suppose that in the binding region, we have $x_1^+(\theta) = x_1^*(\theta)$. Then, we must have that

$$W_1'(x_1^*(\theta), \theta)g(\theta) - \left((1-\delta)G(\theta) - \Gamma(\theta)\right)q(x_1^*(\theta)) - W_1'(x_1^*(\theta), \theta)\Gamma'(\theta) = 0.$$

Moreover, as $W_1'(x_1^*(\theta), \theta) = 0$, we obtain that

$$\Gamma(\theta) = (1-\delta)G(\theta)$$

Let's consider then a multiplier

$$\Gamma(\theta) = \begin{cases} (1-\delta)G(\theta) & \text{if } \theta \in [\underline{\theta}, \theta_\dagger] \\ (1-\delta)G(\theta_\dagger) & \text{if } \theta \in (\theta_\dagger, \bar{\theta}] \end{cases}$$

The directional derivative becomes

$$\nabla\mathcal{L}(\mathbf{x}_1, \mathbf{\Gamma}; \mathbf{h}) = (1-\delta)\left[(1 - G(\theta_\dagger))\bar{\theta}q(x_1^+(\bar{\theta})) + G(\theta_\dagger)\right]h(\bar{\theta}) + \delta\int_{\underline{\theta}}^{\theta_\dagger} W_1'(x_1^+(\theta), \theta)g(\theta)h(\theta)d\theta$$
$$+ \int_{\theta_\dagger}^{\bar{\theta}} \left[W_1'(x_1^+(\theta), \theta)g(\theta) - (1-\delta)\left(G(\theta) - G(\theta_\dagger)\right)q(x_1^+(\theta))\right]h(\theta)d\theta$$

Let

$$\phi(\theta, \mathbf{x}_1^+) \equiv \delta W_1'(x_1^+(\theta), \theta)g(\theta)\mathbf{1}_{\{\theta \le \theta_\dagger\}}$$
$$+ \left[W_1'(x_1^+(\theta), \theta)g(\theta) - (1-\delta)\left(G(\theta) - G(\theta_\dagger)\right)q(x_1^+(\theta))\right]\mathbf{1}_{\{\theta > \theta_\dagger\}}$$

and

$$\Phi(\theta;\mathbf{x}_1^+) \equiv \int_{\underline{\theta}}^{\theta} \phi(\tilde{\theta},\mathbf{x}_1^+)d\tilde{\theta}.$$

The directional derivative can be written as

$$\nabla\mathcal{L}(\mathbf{x}_1^+,\mathbf{\Gamma};\mathbf{h}) = (1-\delta)\left[(1-G(\theta_\dagger))\bar{\theta}q(x_1^+(\bar{\theta}))+G(\theta_\dagger)\right]h(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \Phi'(\theta;\mathbf{x}_1^+)h(\theta)d\theta$$

so after integrating by parts, we obtain that $\nabla\mathcal{L}(\mathbf{x}_1^+,\mathbf{\Gamma};\mathbf{h})$ can be alternatively written as

$$\nabla\mathcal{L}(\mathbf{x}_1,\mathbf{\Gamma};\mathbf{h}) = \left\{\Phi(\bar{\theta};\mathbf{x}_1^+) + (1-\delta)\left[(1-G(\theta_\dagger))\bar{\theta}q(x_1^+(\bar{\theta}))+G(\theta_\dagger)\right]\right\}h(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta;\mathbf{x}_1)dh(\theta)$$

Evaluated at the following policy.

- If $\theta \in [\underline{\theta},\theta_\dagger]$ then $x_1^+(\theta) = x_1^*(\theta)$.
- If $\theta \in (\theta_\dagger,\theta_H]$ then

$$x_1^+(\theta) = \arg\max_{x_1} \left\{W_1(x_1,\theta) + (1-\delta)\frac{G(\theta)-G(\theta_\dagger)}{g(\theta)}V_2(x_1)\right\}$$

- If $\theta \in (\theta_H,\bar{\theta}]$ then $x_1^+(\theta) = x_1^+(\theta_H)$

we get

$$\Phi(\theta;\mathbf{x}_1^+) = \mathbf{1}_{\{\theta>\theta_H\}} \int_{\theta_H}^{\theta} \left[W_1'(x_1^+(\theta_H),\theta)g(\theta) - (1-\delta)\left(G(\theta)-G(\theta_\dagger)\right)q(x_1^+(\theta_H))\right]d\tilde{\theta}.$$

From here, we get that $\{\theta_\dagger,\theta_H\}$ satisfy

$$\int_{\theta_H}^{\bar{\theta}} \left[W_1'(x_1^+(\theta_H),\theta)g(\theta) - (1-\delta)\left(G(\theta)-G(\theta_\dagger)\right)q(x_1^+(\theta_H))\right]d\theta = -(1-\delta)\left[(1-G(\theta_\dagger))\bar{\theta}q(x_1^+(\bar{\theta}))+G(\theta_\dagger)\right]$$

$$W_1(x_1^+(\theta_\dagger),\theta_\dagger) - \theta_H V_2(x_1^+(\theta_H)) + \int_{\theta_\dagger}^{\theta_H} V_2(x_1^+(\tilde{\theta}))d\tilde{\theta} = x_1^+(\theta_H)$$

Finally, we need to verify the saddle point condition for $\mathcal{L}(\mathbf{x}_1,\mathbf{\Gamma})$, which amounts to verifying that $\mathcal{L}(\mathbf{x}_1, \mathbf{\Gamma})$ is concave in $\mathbf{x}_1$. Substituting $\mathbf{\Gamma}$ we get

$$\mathcal{L}(\mathbf{x}_1, \mathbf{\Gamma}) = -(1-\delta)(1-G(\theta_\dagger))\bar{\theta}V_2(x_1^+(\bar{\theta})) + (1-\delta)G(\theta_\dagger)x_1^+(\bar{\theta})$$
$$+ \delta \int_{\underline{\theta}}^{\theta_\dagger} W_1(x_1^+(\theta), \theta)g(\theta)d\theta + \int_{\theta_\dagger}^{\bar{\theta}} \left[ W_1(x_1^+(\theta), \theta) + (1-\delta)\left(\frac{G(\theta) - G(\theta_\dagger)}{g(\theta)}\right) V_2(x_1^+(\theta)) \right] g(\theta)d\theta.$$

Under Assumption 4, we have that $\mathcal{L}(\mathbf{x}_1, \mathbf{\Gamma})$ above is concave in $\mathbf{x}_1$.

$\square$