

Supplemental Appendix to Financial Flexibility under Non-Exclusive Lending

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This supplemental appendix contains supporting proofs for non-exclusive lending and proofs for the exclusive-lending benchmark and the contracting extensions. Numbering in this appendix uses the prefix OA.

OA.1 Supporting Proofs for Non-Exclusive Lending

We use the notation of the paper throughout. In particular, $\varepsilon(x) = -xq'(x)/q(x)$ and $\Gamma(x) = [x\varepsilon'(x) - \varepsilon(x)]/\varepsilon(x)^2$ are the inverse-demand elasticity and the curvature measure defined in Proposition 4, $x^m := \arg \max_{x \in [0, Y]} xq(x)$, and $\rho(\theta) := \mathbb{E}[\theta - \tilde{\theta} \mid \tilde{\theta} \leq \theta]$. Appendix A of the paper states the cutoff condition (28) and collects the properties of these objects used below.

Proof of Lemma 2

Proof. Recall that

$$V_1(x_1, x_0, \theta) = W_1(x_1, \theta) - x_0q(x_1).$$

For part (1), note that, by Lemma 1,

$$\frac{\partial^2 V_1(x_1, x_0, \theta)}{\partial x_1 \partial \theta} = V_2'(x_1) = -q(x_1) \leq 0,$$

with strict inequality for $x_1 < Y$. Thus V_1 has decreasing differences in (x_1, θ) . Since V_1 is concave in x_1 , the monotone-comparative-statics argument of Milgrom and

Shannon (1994) implies that $x_1^f(\theta, x_0)$ is nonincreasing in θ . Similarly, on $x_1 < Y$,

$$\frac{\partial^2 V_1(x_1, x_0, \theta)}{\partial x_1 \partial x_0} = -q'(x_1) > 0,$$

so V_1 has increasing differences in (x_1, x_0) , which implies that $x_1^f(\theta, x_0)$ is nondecreasing in x_0 .

For part (2), $x_1^*(\theta)$ maximizes $W_1(x_1, \theta)$, so

$$\frac{\partial W_1(x_1^*(\theta), \theta)}{\partial x_1} = 0.$$

Evaluating the derivative of V_1 at $x_1 = x_1^*(\theta)$ yields

$$\frac{\partial V_1(x_1^*(\theta), x_0, \theta)}{\partial x_1} = -x_0 q'(x_1^*(\theta)) \geq 0.$$

Since V_1 is concave in x_1 , its maximizer lies at or to the right of $x_1^*(\theta)$, so $x_1^f(\theta, x_0) \geq x_1^*(\theta)$.

For part (3), evaluating the derivative of V_1 at $x_1 = x_0$ gives

$$\frac{\partial V_1(x_0, x_0, \theta)}{\partial x_1} = (1 - \theta)q(x_0) \geq 0,$$

where the expression follows from direct differentiation. Again by concavity, the maximizer lies at or to the right of x_0 , so $x_1^f(\theta, x_0) \geq x_0$. If $\bar{\theta} = 1$, then

$$\frac{\partial V_1(x_0, x_0, \bar{\theta})}{\partial x_1} = 0.$$

Thus x_0 satisfies the first-order condition, and concavity implies $x_1^f(\bar{\theta}, x_0) = x_0$. $\square$

Proof of Proposition 1

Proof. We first characterize the optimal cutoff for a given financing multiplier, then impose the financing constraint. Let $\lambda \geq 0$ be the multiplier and set $\delta = 1/(1 + \lambda)$. Divide the Lagrangian of (15) by $1 + \lambda$. Differentiating its objective with respect to

θ_L and using (14) gives

$$x_1^{f'}(\theta_L) \int_{\underline{\theta}}^{\theta_L} \left[x_0 q'(x_1^f(\theta_L)) + \delta(\theta_L - \theta) q(x_1^f(\theta_L)) \right] dG(\theta).$$

For $\theta_L > \underline{\theta}$, the factors $G(\theta_L)$ and $q(x_1^f(\theta_L))$ are positive, while $x_1^{f'}(\theta_L) < 0$. The derivative therefore has the opposite sign to the left-hand side of the cutoff condition (28). Under Assumptions 3 and 1, this expression increases with θ_L : log-concavity of q and decreasing flexible debt make the first term increasing, and log-concavity of G makes the conditional mean gap nondecreasing. Its limit as θ_L approaches $\underline{\theta}$ is negative because $x_0 > 0$ and $q' < 0$. Thus the objective initially increases and can change from increasing to decreasing only once. If the expression in (28) is nonnegative at $\bar{\theta} = 1$, its unique zero gives the optimum; otherwise, the optimum is $\theta_L = 1$.

Without the financing constraint, $\delta = 1$. Since $x_1^f(1) = x_0$, the expression in (28) at $\theta_L = 1$ has the same sign as

$$x_0 q'(x_0) + (1 - \mathbb{E}[\theta]) q(x_0) = \frac{\partial W_1(x_0, \mathbb{E}[\theta])}{\partial x_1}.$$

By concavity of W_1 , this sign is nonnegative exactly when $x_0 \leq x_1^*(\mathbb{E}[\theta])$. In that case, θ_L^{uc} solves (28) with $\delta = 1$; otherwise, $\theta_L^{\text{uc}} = 1$.

Initial financing increases with the cutoff because x_1^f and q are decreasing. Hence the feasible cutoffs are $[\theta_L^{\text{bc}}, 1]$, where θ_L^{bc} is defined in the proposition. Since the objective with $\delta = 1$ increases up to θ_L^{uc} and decreases thereafter, the constrained optimum is

$$\theta_L = \max\{\theta_L^{\text{uc}}, \theta_L^{\text{bc}}\}.$$

If $\theta_L^{\text{uc}} > \theta_L^{\text{bc}}$, the financing constraint is slack and complementary slackness gives $\delta = 1$.

If $\theta_L^{\text{uc}} \leq \theta_L^{\text{bc}}$, the financing constraint binds. When $\theta_L^{\text{bc}} < 1$, equation (28) gives

$$\delta = - \frac{x_0}{\mathbb{E}[\theta_L^{\text{bc}} - \theta \mid \theta \leq \theta_L^{\text{bc}}]} \frac{q'(x_1^f(\theta_L^{\text{bc}}))}{q(x_1^f(\theta_L^{\text{bc}}))}.$$

This value is positive and at most one because $\theta_L^{\text{bc}} \geq \theta_L^{\text{uc}}$ and the left-hand side of (28), evaluated at $\delta = 1$, increases with the cutoff. When $\theta_L^{\text{bc}} = 1$, the first-order

condition and complementary slackness instead require

$$x_0 \frac{q'(x_0)}{q(x_0)} + \delta \mathbb{E}[1 - \theta] \leq 0.$$

Choosing the largest $\delta \in (0, 1]$ satisfying this inequality gives the multiplier in the proposition. $\square$

OA.1.1 Proof of Proposition 2

Proof. We construct a multiplier that makes the proposed debt limit maximize the Lagrangian, following [Amador, Werning, and Angeletos \(2006\)](#) and [Amador and Bagwell \(2013\)](#). Feasibility and complementary slackness then establish optimality by Theorem 2 in [Luenberger \(1969, p. 221\)](#).

Let us first rewrite the problem in (10). Substituting $W_1(x_1, \theta) = x_0 q(x_1) + V_1(x_1, x_0, \theta)$ together with equation (9) into the objective function in (10) and applying integration by parts, we obtain the following problem:

$$\begin{aligned} W_0(x_0) = \max_{x_1(\theta) \geq x_0} & V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E} \left[x_0 q(x_1(\theta)) - \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta)) \right] \\ \text{subject to} & \\ & V_1(x_1(\theta), x_0, \theta) = V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta})) d\tilde{\theta} \quad \forall \theta \in \Theta \quad (\text{OA.1}) \\ & x_0 \mathbb{E}[q(x_1(\theta))] \geq I \\ & x_1(\theta) \text{ is non-increasing.} \end{aligned}$$

Let P be the set of nonincreasing debt schedules satisfying $x_1(\theta) \geq x_0$. With the financing multiplier $\lambda \geq 0$ from Proposition 1, the relaxation of (OA.1) is

$$\max_{x_1(\theta)} V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E} \left[(1 + \lambda) x_0 q(x_1(\theta)) - \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta)) \right] - \lambda I, \quad (\text{OA.2})$$

subject to $\mathbf{x}_1 \in P$ and the envelope constraint in (OA.1). Dividing by $1 + \lambda$, setting $\delta = 1/(1 + \lambda)$, and omitting the constant term gives

$$\max_{x_1(\theta)} \delta V_1(x_1(\bar{\theta}), x_0, \bar{\theta}) + \mathbb{E} \left[x_0 q(x_1(\theta)) - \delta \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta)) \right], \quad (\text{OA.3})$$

with the same constraints.

Let Λ be the cumulative multiplier on the envelope constraint, with $\Lambda(\underline{\theta}) = 0$ and $\Lambda(\bar{\theta}) = \delta$. Adding that constraint and changing the order of integration gives

$$\begin{aligned} \mathcal{L}(\mathbf{x}_1, \Lambda) &= \int_{\underline{\theta}}^{\bar{\theta}} [x_0 q(x_1(\theta)) g(\theta) + (\Lambda(\theta) - \delta G(\theta)) V_2(x_1(\theta))] d\theta \\ &\quad + \int_{\underline{\theta}}^{\bar{\theta}} V_1(x_1(\theta), x_0, \theta) d\Lambda(\theta). \end{aligned} \tag{OA.4}$$

Write $V_1' = \partial V_1 / \partial x_1$. Since $V_2' = -q$, the directional derivative is

$$\begin{aligned} \nabla \mathcal{L}(\mathbf{x}_1, \Lambda; \mathbf{h}) &= \int_{\underline{\theta}}^{\bar{\theta}} [x_0 q'(x_1(\theta)) g(\theta) - (\Lambda(\theta) - \delta G(\theta)) q(x_1(\theta))] h(\theta) d\theta \\ &\quad + \int_{\underline{\theta}}^{\bar{\theta}} V_1'(x_1(\theta), x_0, \theta) h(\theta) d\Lambda(\theta). \end{aligned} \tag{OA.5}$$

For the proposed schedule $x_1^l(\theta) = \min\{x_1^f(\theta), x_1^f(\theta_L)\}$, choose

$$\Lambda(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, \theta_L), \\ \delta G(\theta) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} g(\theta), & \theta \in [\theta_L, \bar{\theta}), \\ \delta, & \theta = \bar{\theta}. \end{cases} \tag{OA.6}$$

For $\theta \geq \theta_L$, the flexible-debt condition gives $V_1'(x_1^l(\theta), x_0, \theta) = 0$. Thus the multiplier cancels the derivative above the cutoff, and its jumps at θ_L and $\bar{\theta}$ contribute zero. Below the cutoff, $\Lambda = 0$. Define

$$\Phi(\theta) = x_0 q'(x_1^f(\theta_L)) G(\theta) + \delta q(x_1^f(\theta_L)) \int_{\underline{\theta}}^{\theta} G(\tilde{\theta}) d\tilde{\theta}.$$

Equation (OA.5) and integration by parts then give

$$\nabla \mathcal{L}(\mathbf{x}_1^l, \Lambda; \mathbf{h}) = \Phi(\theta_L) h(\theta_L) - \int_{\underline{\theta}}^{\theta_L} \Phi(\theta) dh(\theta).$$

By Lemma 3, $\int_{\underline{\theta}}^{\theta} G(\tilde{\theta}) d\tilde{\theta} / G(\theta) = \mathbb{E}[\theta - \tilde{\theta} \mid \tilde{\theta} \leq \theta]$ is nondecreasing in θ . The cutoff

condition in Proposition 1 therefore implies

$$x_0 \frac{q'(x_1^f(\theta_L))}{q(x_1^f(\theta_L))} + \delta \mathbb{E}[\theta - \tilde{\theta} \mid \tilde{\theta} \leq \theta] \leq 0, \quad \underline{\theta} < \theta \leq \theta_L. \quad (\text{OA.7})$$

Hence $\Phi(\theta) \leq 0$ on $[\underline{\theta}, \theta_L]$, with $\Phi(\underline{\theta}) = 0$ by definition. If $\theta_L < \bar{\theta}$, the cutoff condition is an equality, so $\Phi(\theta_L) = 0$. If $\theta_L = \bar{\theta} = 1$, it instead gives $\Phi(1) \leq 0$ and the proposed schedule is constant at x_0 .

For any $\tilde{\mathbf{x}}_1 \in P$, let $\mathbf{h} = \tilde{\mathbf{x}}_1 - \mathbf{x}_1^l$. Because $\mathbf{x}_1^l$ is constant below the cutoff, $dh \leq 0$ there. The integral term in the directional derivative is therefore nonpositive. The first term is zero if $\theta_L < 1$ and nonpositive if $\theta_L = 1$, since $\Phi(1) \leq 0 \leq h(1)$. Thus

$$\nabla \mathcal{L}(\mathbf{x}_1^l, \Lambda; \tilde{\mathbf{x}}_1 - \mathbf{x}_1^l) \leq 0 \quad \text{for every } \tilde{\mathbf{x}}_1 \in P.$$

It remains to verify concavity. Substituting (OA.6) into (OA.4) gives

$$\begin{aligned} \mathcal{L}(\mathbf{x}_1, \Lambda) = & \int_{\underline{\theta}}^{\theta_L} \left[x_0 q(x_1(\theta)) - \delta \frac{G(\theta)}{g(\theta)} V_2(x_1(\theta)) \right] g(\theta) d\theta \\ & + \int_{\theta_L}^{\bar{\theta}} \left[x_0 q(x_1(\theta)) + \frac{x_0 q'(x_1^f(\theta))}{q(x_1^f(\theta))} V_2(x_1(\theta)) \right] g(\theta) d\theta \\ & + \int_{\underline{\theta}}^{\bar{\theta}} V_1(x_1(\theta), x_0, \theta) d\Lambda(\theta). \end{aligned} \quad (\text{OA.8})$$

Assumption 3 makes q and V_1 concave. Since $V_2'' = -q' > 0$ and its coefficients in the first two integrals are nonpositive, those integrals are concave as well. It therefore suffices to show that Λ is nondecreasing.

When $\theta_L = 1$, Λ is zero below one and jumps to δ at one. When $\theta_L < 1$, condition (18) makes Λ nondecreasing on $[\theta_L, 1)$. At the cutoff, Proposition 1 gives

$$\Lambda(\theta_L) = \delta G(\theta_L) \left[1 - \mathbb{E}[\theta_L - \theta \mid \theta \leq \theta_L] \frac{g(\theta_L)}{G(\theta_L)} \right] \geq 0.$$

The bracket is $\rho'(\theta_L) \geq 0$ by Lemma 3. Finally, $q' < 0$ gives $\Lambda(1-) \leq \delta = \Lambda(1)$. Thus both jumps are nonnegative, and $\mathcal{L}$ is concave.

The directional-derivative inequality now implies that $\mathbf{x}_1^l$ maximizes the Lagrangian over P (Luenberger, 1969, Lemma 1, p. 227). The proposed schedule satisfies the envelope constraint and the financing constraint, and its financing

multiplier satisfies complementary slackness by Proposition 1. It therefore solves the original problem. $\square$

OA.2 Exclusive Lending

For notational convenience, let us define

$$\bar{V}_1 = \tau_1(\bar{\theta}) + \bar{\theta}V_2(x_1(\bar{\theta}))$$

as type $\bar{\theta}$'s continuation payoff at $t = 1$.

Lemma OA.1. *Any incentive-compatible pair $\{\tau_1(\theta), x_1(\theta)\}$ satisfying $\bar{V}_1 \geq \bar{\theta}V_2(x_1(\bar{\theta}))$ satisfies the constraint*

$$\tau_1(\theta) = \bar{V}_1 - \theta V_2(x_1(\theta)) - \int_{\theta}^{\bar{\theta}} V_2(x_1(\tilde{\theta})) d\tilde{\theta} \geq 0 \quad \forall \theta \in \Theta.$$

Proof. For any $\theta'' > \theta'$, incentive compatibility for type θ' implies

$$\tau_1(\theta') - \tau_1(\theta'') \geq \theta' [V_2(x_1(\theta'')) - V_2(x_1(\theta'))] \geq 0,$$

where the second inequality follows because both x_1 and V_2 are non-increasing. Thus, τ_1 is non-increasing, including when the debt schedule has jumps. It is therefore sufficient to impose $\tau_1(\bar{\theta}) \geq 0$, which is equivalent to $\bar{V}_1 \geq \bar{\theta}V_2(x_1(\bar{\theta}))$. $\square$

Proof of Proposition 6

Proof. We first characterize the cutoff above which debt is constant, then verify optimality of the partially separating and fully pooled schedules.

Fix δ and let P be the set of non-increasing schedules $\mathbf{x}_1 = \{x_1(\theta)\}_{\theta \in \Theta}$. Write the objective in (26) as

$$\mathcal{U}(\mathbf{x}_1) = \int_{\theta}^{\bar{\theta}} W_1(x_1(\theta), \phi_{\delta}(\theta)) dG(\theta) - (1 - \delta)\bar{\theta}V_2(x_1(\bar{\theta})).$$

To express the optimality conditions, define

$$\Phi(\theta; \mathbf{x}_1) = \int_{\theta}^{\bar{\theta}} \left[\frac{\partial W_1(x_1(\tilde{\theta}), \tilde{\theta})}{\partial x_1} g(\tilde{\theta}) - (1 - \delta)G(\tilde{\theta})q(x_1(\tilde{\theta})) \right] d\tilde{\theta}.$$

Since $V_2' = -q$, integration by parts gives the directional derivative

$$\nabla \mathcal{U}(\mathbf{x}_1; \mathbf{h}) = [\Phi(\bar{\theta}; \mathbf{x}_1) + (1 - \delta)\bar{\theta}q(x_1(\bar{\theta}))] h(\bar{\theta}) - \int_{\theta}^{\bar{\theta}} \Phi(\theta; \mathbf{x}_1) dh(\theta).$$

The objective is concave by Assumption 5 and the convexity of V_2 . The necessary and sufficient conditions are therefore (Luenberger, 1969, Lemma 1, p. 227)

$$\nabla \mathcal{U}(\mathbf{x}_1; \tilde{\mathbf{x}}_1) \leq 0 \quad \text{for every } \tilde{\mathbf{x}}_1 \in P, \quad (\text{OA.9a})$$

$$\nabla \mathcal{U}(\mathbf{x}_1; \mathbf{x}_1) = 0. \quad (\text{OA.9b})$$

The cutoff. Let $x_1^s(\theta) = x_1^*(\phi_\delta(\theta))$. This schedule is non-increasing and, whenever $\phi_\delta(\theta) < 1$, satisfies

$$\frac{\partial W_1(x_1^s(\theta), \theta)}{\partial x_1} = (1 - \delta) \frac{G(\theta)}{g(\theta)} q(x_1^s(\theta)). \quad (\text{OA.10})$$

Consider the schedule $x_1(\theta) = \max\{x_1^s(\theta), x_1^s(\theta_H)\}$. Then $\Phi(\theta; \mathbf{x}_1) = 0$ for $\theta \leq \theta_H$, provided $\phi_\delta(\theta_H) < 1$, which we verify below. Choose θ_H to satisfy

$$\int_{\theta_H}^{\bar{\theta}} \left[\frac{\partial W_1(x_1^s(\theta_H), \tilde{\theta})}{\partial x_1} g(\tilde{\theta}) - (1 - \delta)G(\tilde{\theta})q(x_1^s(\theta_H)) \right] d\tilde{\theta} + (1 - \delta)\bar{\theta}q(x_1^s(\theta_H)) = 0. \quad (\text{OA.11})$$

Integrating by parts gives the cutoff equation stated in the proposition. To show that it has a unique solution, combine $\partial^2 W_1 / \partial \theta \partial x_1 = -q$ and the pointwise condition above to obtain

$$\frac{\partial W_1(x_1^s(\theta_H), \theta)}{\partial x_1} = q(x_1^s(\theta_H)) \left[\theta_H - \theta + (1 - \delta) \frac{G(\theta_H)}{g(\theta_H)} \right].$$

Substitute this expression into (OA.11) and divide by $q(x_1^s(\theta_H)) > 0$. Integration by parts then gives

$$H(\theta_H) := (1 - \delta) \frac{G(\theta_H)(1 - G(\theta_H))}{g(\theta_H)} + \theta_H - \delta \left[\bar{\theta} - \int_{\theta_H}^{\bar{\theta}} G(\tilde{\theta}) d\tilde{\theta} \right] = 0.$$

At the lowest and highest types,

$$H(\underline{\theta}) = \underline{\theta} - \delta \mathbb{E}[\theta], \quad H(\bar{\theta}) = (1 - \delta)\bar{\theta}.$$

The derivative is

$$H'(\theta_H) = (1 - G(\theta_H)) \left[1 + (1 - \delta) \frac{d}{d\theta_H} \frac{G(\theta_H)}{g(\theta_H)} \right] > 0 \quad \text{for } \underline{\theta} < \theta_H < \bar{\theta},$$

because log-concavity of G makes G/g non-decreasing. Thus, a unique cutoff exists if and only if $\delta \geq \underline{\theta}/\mathbb{E}[\theta]$. At equality, $\theta_H = \underline{\theta}$ and the schedule is fully pooled; at $\delta = 1$, $\theta_H = \bar{\theta}$ and the schedule is efficient.

For $\underline{\theta}/\mathbb{E}[\theta] \leq \delta < 1$, the root satisfies $\theta_H < \bar{\theta}$. Rearranging $H(\theta_H) = 0$ gives

$$\phi_\delta(\theta_H) = \delta \mathbb{E}[\theta \mid \theta \geq \theta_H] - (1 - \delta) \frac{\theta_H G(\theta_H)}{1 - G(\theta_H)} \leq \delta \bar{\theta} < 1.$$

Since ϕ_δ is non-decreasing, this justifies the use of the first-order condition (OA.10) on $[\underline{\theta}, \theta_H]$. At $\delta = 1$, the terminal term vanishes and the efficient schedule maximizes the objective pointwise.

Verification. For $\underline{\theta}/\mathbb{E}[\theta] \leq \delta < 1$, by (OA.11), the coefficient of $h(\bar{\theta})$ in the directional derivative is zero for the proposed schedule. For $\theta \geq \theta_H$, we have $x_1^s(\theta_H) \geq x_1^s(\theta)$. Concavity of the virtual payoff therefore implies

$$\frac{\partial W_1(x_1^s(\theta_H), \theta)}{\partial x_1} g(\theta) - (1 - \delta) G(\theta) q(x_1^s(\theta_H)) \leq 0.$$

Hence, $\Phi(\theta; \mathbf{x}_1) \leq 0$. Every non-increasing $\tilde{\mathbf{x}}_1$ then satisfies

$$\nabla \mathcal{U}(\mathbf{x}_1; \tilde{\mathbf{x}}_1) = - \int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta; \mathbf{x}_1) d\tilde{x}_1(\theta) \leq 0.$$

For $\tilde{\mathbf{x}}_1 = \mathbf{x}_1$, the integral is zero: $\Phi = 0$ below θ_H , and x_1 is constant above it. Both conditions in (OA.9) hold.

When $\delta < \underline{\theta}/\mathbb{E}[\theta]$, consider the constant schedule $x_1(\theta) = \bar{x}_1 := x_1^*(\delta \mathbb{E}[\theta])$ for every type. Integration by parts gives

$$\Phi(\bar{\theta}; \mathbf{x}_1) + (1 - \delta) \bar{\theta} q(\bar{x}_1) = \bar{x}_1 q'(\bar{x}_1) + (1 - \delta \mathbb{E}[\theta]) q(\bar{x}_1) = 0.$$

Since x_1^* is decreasing and $\delta\mathbb{E}[\theta] < \underline{\theta}$, we have $\bar{x}_1 > x_1^s(\underline{\theta}) \geq x_1^s(\theta)$. The same concavity argument gives $\Phi(\theta; \mathbf{x}_1) \leq 0$, so (OA.9a) holds. Because the schedule is constant, it also satisfies (OA.9b), so full pooling is optimal. $\square$

Proof of Proposition 7

Proof. We characterize the three investment regimes by determining how much initial financing each schedule in Proposition 6 can raise.

For each $\delta \in [0, 1]$, let $x_1^{s*}(\theta; \delta)$ be the optimal schedule in Proposition 6. Set $\tau_1^{s*}(\bar{\theta}; \delta) = 0$ and recover the remaining transfers from (22). By (23), initial pledgeable income is

$$\begin{aligned} \mathcal{T}_0(\delta) &:= \mathbb{E}[x_1^{s*}(\theta; \delta)q(x_1^{s*}(\theta; \delta)) - \tau_1^{s*}(\theta; \delta)] \\ &= \mathbb{E}\left[W_1(x_1^{s*}(\theta; \delta), \theta) + \frac{G(\theta)}{g(\theta)}V_2(x_1^{s*}(\theta; \delta))\right] - \bar{\theta}V_2(x_1^{s*}(\bar{\theta}; \delta)). \end{aligned} \quad (\text{OA.12})$$

Pledgeable income. For a feasible schedule x_1 , write $\mathcal{S}(x_1) = \mathbb{E}[W_1(x_1(\theta), \theta)]$ and let $\mathcal{T}_0(x_1)$ denote its pledgeable income. The objective is $\delta\mathcal{S}(x_1) + (1 - \delta)\mathcal{T}_0(x_1)$. For $\delta > 0$, this is equivalent to maximizing $\mathcal{S}(x_1) + \lambda\mathcal{T}_0(x_1)$, where $\lambda = (1 - \delta)/\delta$. If $\lambda' > \lambda$ and x'_1, x_1 are the respective optimal schedules, revealed preference gives

$$\begin{aligned} \mathcal{S}(x_1) + \lambda\mathcal{T}_0(x_1) &\geq \mathcal{S}(x'_1) + \lambda\mathcal{T}_0(x'_1), \\ \mathcal{S}(x'_1) + \lambda'\mathcal{T}_0(x'_1) &\geq \mathcal{S}(x_1) + \lambda'\mathcal{T}_0(x_1). \end{aligned}$$

Adding yields $(\lambda' - \lambda)[\mathcal{T}_0(x'_1) - \mathcal{T}_0(x_1)] \geq 0$. Thus, pledgeable income is weakly increasing in λ and weakly decreasing in δ .

Let $\delta^p = \underline{\theta}/\mathbb{E}[\theta]$. For $\delta \geq \delta^p$, the cutoff is continuous in δ because H is continuous and strictly increasing in the cutoff. For $\delta \leq \delta^p$, the optimal schedule is $x_1^*(\delta\mathbb{E}[\theta])$. The two schedules agree at δ^p , where $\theta_H = \underline{\theta}$. Continuity of x_1^* therefore makes the optimal schedule continuous in δ . The schedules are bounded. Dominated convergence therefore applies in (OA.12), so $\mathcal{T}_0$ is continuous on $[0, 1]$.

At $\delta = 1$, the schedule is efficient. Under full pooling, the envelope condition and the zero transfer to the highest type imply zero transfers to every type. Thus, at δ^p the borrower pledges $x_1^*(\underline{\theta})q(x_1^*(\underline{\theta}))$, and at $\delta = 0$ the schedule maximizes revenue at

$x_1^*(0)$. These observations give

$$\mathcal{T}_0(1) = I^s, \quad \mathcal{T}_0(\delta^p) = I^p, \quad \mathcal{T}_0(0) = I^{\max}.$$

Here, $\delta = 0$ is understood as the limit as $\lambda \rightarrow \infty$.

Investment regimes. If $I \leq I^s$, the efficient schedule is feasible and optimal, with $\delta^{s*} = 1$. The financing constraint is slack when $I < I^s$ and binds with a zero multiplier when $I = I^s$.

If $I^s < I < I^p$, continuity gives $\delta^{s*} \in (\delta^p, 1)$ with $\mathcal{T}_0(\delta^{s*}) = I$. Proposition 6 then gives the partially separating schedule. The pair (θ_H, δ^{s*}) satisfies (27) and $\tau_0 = I$; this argument establishes existence without requiring uniqueness of the pair.

If $I^p \leq I \leq I^{\max}$, continuity gives $\delta^{s*} \in [0, \delta^p]$. All types receive the same debt level $x_1^*(\delta^{s*}\mathbb{E}[\theta])$ and zero transfers. The financing constraint becomes

$$x_1^*(\delta^{s*}\mathbb{E}[\theta])q(x_1^*(\delta^{s*}\mathbb{E}[\theta])) = I.$$

To prove uniqueness in this region, let $a = \delta\mathbb{E}[\theta]$. The first-order condition for $x_1^*(a)$ is

$$x_1^*(a)q'(x_1^*(a)) + (1-a)q(x_1^*(a)) = 0,$$

and $x_1^{s*}(a) < 0$ by Lemma 3. The first-order condition also gives $q(x_1^*(a)) + x_1^*(a)q'(x_1^*(a)) = aq(x_1^*(a))$. Hence, in the pooled region,

$$\mathcal{T}'_0(\delta) = \mathbb{E}[\theta]aq(x_1^*(a))x_1^{s*}(a) < 0 \quad \text{for } \delta > 0.$$

Together with continuity at zero, this strict decrease proves uniqueness. The values of $\mathcal{T}_0$ at δ^p and zero give $\delta^{s*} = \delta^p$ when $I = I^p$ and $\delta^{s*} = 0$ when $I = I^{\max}$. $\square$

Let $\{x_1^{s*}, \tau_1^{s*}\}$ be the exclusive allocation characterized in Propositions 6 and 7, with discount factor δ^{s*} , cutoff θ_H (equal to $\bar{\theta}$ under full pooling), and $x_0^s := x_1^{s*}(\bar{\theta})$. The implementation results below rest on the same properties of this allocation, which the next lemma collects. Define the exclusive lender's net repayment in state θ as

$$L^s(\theta) := x_1^{s*}(\theta)q(x_1^{s*}(\theta)) - \tau_1^{s*}(\theta).$$

Lemma OA.2 (Properties of the exclusive allocation).

1. On $[\theta, \theta_H]$, $\tau_1^{s*'}(\theta) = \theta q(x_1^{s*}(\theta)) x_1^{s*'}(\theta)$ and

$$L^{s'}(\theta) = (1 - \delta^{s*}) \frac{G(\theta)}{g(\theta)} q(x_1^{s*}(\theta)) x_1^{s*'}(\theta) \leq 0.$$

On $[\theta_H, \bar{\theta}]$, $\tau_1^{s*}(\theta) = 0$ and $L^s(\theta) = x_0^s q(x_0^s)$.

2. $\mathbb{E}[L^s(\theta)] = \tau_0^{s*} \geq \max\{I, x_0^s q(x_0^s)\}$. If the financing constraint binds, then $x_0^s \leq x_0^{\min}$.

3. $0 \leq \tau_1^{s*}(\theta) \leq (x_1^{s*}(\theta) - x_0^s) q(x_1^{s*}(\theta))$ for every $\theta \in \Theta$, with strict upper inequality for $\theta < \theta_H$.

4. Consider a $t = 1$ contract that offers actions indexed by $\hat{\theta} \in \Theta$, where action $\hat{\theta}$ leaves the borrower with total debt $x_1^{s*}(\hat{\theta})$ and net cash $\tau_1^{s*}(\hat{\theta})$. Then type θ 's payoff from action $\hat{\theta}$ is $\tau_1^{s*}(\hat{\theta}) + \theta V_2(x_1^{s*}(\hat{\theta}))$, the truthful action is optimal, and the contract reproduces the exclusive allocation. If existing lenders' expected payoff under truthful actions equals $\mathbb{E}[L^s(\theta)]$, the contract raises $\tau_0^{s*} \geq I$ at $t = 0$.

Proof. For part (1), differentiate the envelope condition (22) and use $V_2' = -q$ to get $\tau_1^{s*'} = \theta q(x_1^{s*}) x_1^{s*'}$. Hence

$$L^{s'}(\theta) = [q(x_1^{s*}) + x_1^{s*} q'(x_1^{s*}) - \theta q(x_1^{s*})] x_1^{s*'} = \frac{\partial W_1(x_1^{s*}(\theta), \theta)}{\partial x_1} x_1^{s*'}(\theta),$$

and on the separating region the first-order condition in the proof of Proposition 6 gives $\partial W_1 / \partial x_1 = (1 - \delta^{s*})(G/g)q$. The sign follows from $x_1^{s*'} < 0$. Above θ_H , debt is constant at x_0^s and $\tau_1^{s*}(\bar{\theta}) = 0$, so the envelope condition gives $\tau_1^{s*} = 0$ and $L^s = x_0^s q(x_0^s)$.

For part (2), binding participation in (23) gives $\tau_0^{s*} = \mathbb{E}[L^s(\theta)]$, and the financing constraint gives $\tau_0^{s*} \geq I$. By part (1), L^s is nonincreasing, so $\mathbb{E}[L^s] \geq L^s(\bar{\theta}) = x_0^s q(x_0^s)$. If financing binds, then $x_0^s q(x_0^s) \leq I$. By Proposition 6, $x_0^s = x_1^*(a)$ for some $a \geq 0$, so $x_0^s \leq x_1^*(0) = x^m$, and $xq(x)$ increases on $[0, x^m]$ by Lemma 3. Hence $x_0^s \leq x_0^{\min}$.

For part (3), the lower bound is Lemma OA.1 with $\tau_1^{s*}(\bar{\theta}) = 0$. Let $B(\theta) := L^s(\theta) - x_0^s q(x_1^{s*}(\theta))$, so $B(\theta_H) = 0$ and, by part (1),

$$B'(\theta) = \left[(1 - \delta^{s*}) \frac{G(\theta)}{g(\theta)} q(x_1^{s*}(\theta)) - x_0^s q'(x_1^{s*}(\theta)) \right] x_1^{s*'}(\theta) \leq 0$$

on the separating region, with strict inequality unless $\delta^{s*} = 1$ and $x_0^s = 0$. In that case L^s is constant at $x_0^s q(x_0^s) = 0$, so $\tau_0^{s*} = 0 < I$, contradicting part (2). Hence $B(\theta) > 0$ for $\theta < \theta_H$, which is the strict upper bound. On the pooled region both sides of the bound are zero.

For part (4), the borrower's payoff is its net cash plus θ times the residual claim at the resulting total debt. Optimality of the truthful action is the incentive constraint (21) of the exclusive allocation. The financing claim is part (2). $\square$

Proof of Proposition 9

Proof. Under full pooling, Lemma OA.2 gives $\tau_1^{s*}(\theta) = 0$ for every type, so there is no additional borrowing to price. Now suppose $\theta_H > \underline{\theta}$. On $[\theta_H, \bar{\theta}]$, debt is constant at x_0^s and transfers are zero. On the separating region, Lemma OA.2 gives $\tau_1^{s*'}(\theta) = \theta q(x_1^{s*}(\theta)) x_1^{s*'}(\theta)$ and

$$0 \leq \tau_1^{s*}(\theta) < (x_1^{s*}(\theta) - x_0^s) q(x_1^{s*}(\theta)) \quad \text{for } \theta < \theta_H.$$

To recover the price schedule, let $\theta^s(\Delta x_1)$ be the inverse of $\Delta x_1^s(\theta)$ on the separating region and write $\tilde{\tau}_1^s(\Delta x_1) = \tau_1^{s*}(\theta^s(\Delta x_1))$. The envelope derivative above and the chain rule give

$$\frac{d\tilde{\tau}_1^s(\Delta x_1)}{d\Delta x_1} = \theta^s(\Delta x_1) q(x_0^s + \Delta x_1).$$

Since $\tilde{\tau}_1^s(0) = 0$, integration yields

$$\tilde{\tau}_1^s(\Delta x_1) = \int_0^{\Delta x_1} \theta^s(z) q(x_0^s + z) dz.$$

For positive borrowing, dividing by Δx_1 gives the price p_1^s in the proposition and reproduces the transfer $\tau_1^{s*}(\theta) = \Delta x_1^s(\theta) p_1^s(\Delta x_1^s(\theta))$. Define the price at zero by continuity; the transfer is then zero when no additional debt is issued. Finally, the strict upper bound gives $p_1^s(\Delta x_1) < q(x_0^s + \Delta x_1)$ for every positive draw on the separating region. $\square$

OA.3 Comparison of Exclusive and Non-Exclusive Lending

Proof of Proposition 8

Proof. We first compare initial debt, then show that the debt schedules cross once. Write $x_s(\theta) = x_1^{s*}(\theta)$ and $x_n(\theta) = x_1^{n*}(\theta)$ within this proof.

Initial debt. Under Assumption 4, $I^s = 0$, so the financing constraint binds and Lemma OA.2 gives $x_0^s \leq x_0^{\min}$. Non-exclusive feasibility gives $x_0^{n*} \geq x_0^{\min}$, and Proposition 3 makes this inequality strict when $\theta_L^{n*} < \bar{\theta}$. Thus,

$$x_0^s \leq x_0^{\min} < x_0^{n*}.$$

The crossing. Recall from Lemma 3 that ε is strictly increasing. Let δ^{n*} be the value of δ in the optimal non-exclusive debt-limit problem. The cutoff condition holds at $\theta_L^{n*} \in (\underline{\theta}, \bar{\theta})$ with $\delta^{n*} \leq 1$, so the same lemma gives

$$x_n(\underline{\theta}) = x_L^{n*} < x_1^*(\underline{\theta}) = x_s(\underline{\theta}).$$

On $[\underline{\theta}, \min\{\theta_H^{s*}, \theta_L^{n*}\}]$, non-exclusive debt is constant at x_L^{n*} and exclusive debt strictly decreases. Their difference therefore increases strictly. To establish a crossing, we need only show that non-exclusive debt is higher at the smaller cutoff.

If $\theta_L^{n*} \leq \theta_H^{s*}$, the first-order conditions at θ_L^{n*} give

$$\varepsilon(x_L^{n*}) - \varepsilon(x_s(\theta_L^{n*})) = \frac{x_0^{n*}}{x_L^{n*}} \varepsilon(x_L^{n*}) + (1 - \delta^{s*}) \frac{G(\theta_L^{n*})}{g(\theta_L^{n*})} > 0.$$

If $\theta_H^{s*} < \theta_L^{n*}$, the initial-debt comparison gives

$$x_n(\theta_H^{s*}) = x_L^{n*} \geq x_0^{n*} > x_0^s = x_s(\theta_H^{s*}).$$

Thus, continuity and strict monotonicity give a unique crossing $\theta^\dagger \in (\underline{\theta}, \min\{\theta_H^{s*}, \theta_L^{n*}\})$, with non-exclusive debt lower before the crossing and higher after it on this interval.

It remains to rule out another crossing. If $\theta_L^{n*} \leq \theta < \theta_H^{s*}$, the two separating

first-order conditions give

$$\varepsilon(x_n(\theta)) - \varepsilon(x_s(\theta)) = \frac{x_0^{n*}}{x_n(\theta)} \varepsilon(x_n(\theta)) + (1 - \delta^{s*}) \frac{G(\theta)}{g(\theta)} > 0,$$

so $x_n(\theta) > x_s(\theta)$. For $\theta \geq \theta_H^{s*}$, exclusive debt is x_0^s , while $x_n(\theta) \geq x_0^{n*} > x_0^s$. Hence, $x_n(\theta) > x_s(\theta)$ for every $\theta > \theta^\dagger$. $\square$

OA.4 Richer Contracting Mechanisms

OA.4.1 Performance-Sensitive Existing Debt

Proof of Proposition 10

Proof. We show that new issuance identifies the borrower's type on the separating region and that the resulting debt adjustment reproduces the exclusive allocation.

Write $x(\theta) = x_1^{s*}(\theta)$ and $\tau(\theta) = \tau_1^{s*}(\theta)$ for the exclusive allocation in Section 4, and set $x_0^s = x(\bar{\theta})$. Lemma OA.2 gives $0 \leq \tau(\theta) \leq (x(\theta) - x_0^s)q(x(\theta))$. Since $q(x(\theta)) > 0$, define

$$x_0^{\text{ps}}(\theta) = x(\theta) - \frac{\tau(\theta)}{q(x(\theta))}, \quad \Delta x_1^{\text{ps}}(\theta) = \frac{\tau(\theta)}{q(x(\theta))}.$$

The transfer bound implies $x_0^s \leq x_0^{\text{ps}}(\theta) \leq x(\theta)$. Thus, the adjustment to existing debt is nonnegative, and new issuance finances the transfer while leaving total debt equal to $x(\theta)$.

On the separating region, Lemma OA.2 gives $\tau'(\theta) = \theta q(x(\theta))x'(\theta)$. Differentiating new issuance therefore yields

$$\frac{d}{d\theta} \Delta x_1^{\text{ps}}(\theta) = \frac{x'(\theta)}{q(x(\theta))^2} [\theta q(x(\theta))^2 - \tau(\theta)q'(x(\theta))] < 0,$$

because $x' < 0$, $q' < 0$, $\theta > 0$, and $\tau \geq 0$. Hence, the issuance schedule is invertible on that region. On the pooled region, $\tau = 0$ and new issuance is zero.

Let $\theta^{\text{ps}}(\Delta x_1) = (\Delta x_1^{\text{ps}})^{-1}(\Delta x_1)$. Set initial financing to τ_0^{s*} and, after an issue of face value Δx_1 , adjust the existing claim from its baseline x_0^s to

$$x_0^{\text{ps}}(\Delta x_1) = x(\theta^{\text{ps}}(\Delta x_1)) - \Delta x_1.$$

The issue prescribed for report $\hat{\theta}$ leaves the borrower with total debt $x(\hat{\theta})$ and, because

competitive new lenders pay $\Delta x_1^{\text{ps}}(\hat{\theta})q(x(\hat{\theta})) = \tau(\hat{\theta})$, with net cash $\tau(\hat{\theta})$. Existing lenders receive $x_0^{\text{ps}}(\theta)q(x(\theta)) = x(\theta)q(x(\theta)) - \tau(\theta)$ in state θ . Lemma OA.2 therefore gives incentive compatibility among the prescribed issues, replication of the exclusive allocation, and feasibility of initial financing. $\square$

OA.4.2 Contingent Prepayment

We now show that a nonlinear debt sweep can reproduce the exclusive-lending allocation under non-exclusive lending. Let $x_0^s := x_1^{s*}(\bar{\theta})$ denote the existing debt level from the exclusive benchmark. If type θ is to end $t = 1$ with total debt $x_1^{s*}(\theta)$, then a prepayment of $b(\theta)$ requires gross new issuance of face value

$$\Delta x_1^{\text{ds}}(\theta) := x_1^{s*}(\theta) - x_0^s + b(\theta).$$

The borrower receives the proceeds from this new issue and uses $b(\theta)$ to repay existing debt. To reproduce the exclusive-lending transfer, define

$$b(\theta) := \frac{(x_1^{s*}(\theta) - x_0^s)q(x_1^{s*}(\theta)) - \tau_1^{s*}(\theta)}{1 - q(x_1^{s*}(\theta))}. \quad (\text{OA.13})$$

The formula pins down the cash-flow transfer needed to reproduce the exclusive-lending payoff, but not every such transfer is a feasible sweep: the contract can accelerate only debt that is already outstanding. The bound $b(\theta) \leq x_0^s$ is therefore an implementation condition rather than an incentive constraint. If it fails, the schedule asks the borrower to prepay more legacy debt than exists, so it is not a debt sweep.

The next proposition shows that a nonlinear debt sweep reproduces the exclusive-lending allocation for every issuance level on the equilibrium path.¹

Proposition OA.1 (On-Path Replication with a Nonlinear Debt Sweep). *Suppose the exclusive benchmark is not fully pooled. Assume the endpoint condition that ensures the sweep never requires prepayment above existing debt, $b(\theta) \leq x_0^s$:*

$$x_1^{s*}(\theta)q(x_1^{s*}(\theta)) - \tau_1^{s*}(\theta) \leq x_0^s. \quad (\text{OA.14})$$

The issuance schedule $\Delta x_1^{\text{ds}}(\theta)$ is strictly decreasing on the separating region and

¹The issuance restriction in footnote 7 also applies to the debt-sweep schedule, subject to the prepayment feasibility bounds.

therefore invertible. Define

$$\theta^{\text{ds}}(\Delta x_1) := (\Delta x_1^{\text{ds}})^{-1}(\Delta x_1).$$

After an issue of face value Δx_1 , the debt-sweep prepayment is

$$b^{\text{ds}}(\Delta x_1) := b(\theta^{\text{ds}}(\Delta x_1)).$$

For every issuance level prescribed by Δx_1^{ds} , the debt sweep requires the borrower to use the fraction

$$\alpha^{\text{ds}}(\Delta x_1) = \frac{b^{\text{ds}}(\Delta x_1)}{q(x_0^s + \Delta x_1 - b^{\text{ds}}(\Delta x_1)) \Delta x_1}$$

of the cash proceeds from new debt issuance to prepay existing debt. This rule reproduces the corresponding post-prepayment debt, interim transfer, and continuation payoff. No type gains by choosing the issuance level prescribed for another type. Hence, the rule replicates the exclusive-lending allocation along the prescribed issuance path.

Proof of Proposition OA.1

Proof. The sweep reproduces the exclusive payoff by leaving the borrower with the same net cash and total debt. We verify that the required prepayment is feasible and can be specified as a function of gross issuance.

Write $x(\theta) = x_1^{s*}(\theta)$, $\tau(\theta) = \tau_1^{s*}(\theta)$, and $x_0 = x_0^s$. If a type θ chooses the issue prescribed for report $\hat{\theta}$, gross issuance is $x(\hat{\theta}) - x_0 + b(\hat{\theta})$ and debt after prepayment is $x(\hat{\theta})$. By (OA.13), net proceeds are

$$(x(\hat{\theta}) - x_0 + b(\hat{\theta})) q(x(\hat{\theta})) - b(\hat{\theta}) = \tau(\hat{\theta}).$$

Thus, the issue prescribed for $\hat{\theta}$ leaves the borrower with total debt $x(\hat{\theta})$ and net cash $\tau(\hat{\theta})$, and Lemma OA.2 gives incentive compatibility among the prescribed issuance levels.

Prepayment feasibility. Since $q(x) < 1$, the denominator in (OA.13) is positive, and the transfer bound in Lemma OA.2 gives $b(\theta) \geq 0$. The condition $b(\theta) \leq x_0$ is equivalent to $L^s(\theta) \leq x_0$, where L^s is the exclusive lender's net repayment. By

Lemma OA.2, L^s is nonincreasing, so Condition (OA.14) ensures $b(\theta) \leq x_0$ for every type.

Issuance and financing. Differentiating $\Delta x_1^{\text{ds}}(\theta) = x(\theta) + b(\theta) - x_0$ and using (OA.13) gives

$$\frac{d}{d\theta} \Delta x_1^{\text{ds}}(\theta) = \frac{x'(\theta)}{1 - q(x(\theta))} [1 - \theta q(x(\theta)) + q'(x(\theta)) \Delta x_1^{\text{ds}}(\theta)].$$

Because $b(\theta) \leq x_0$, gross issuance satisfies $\Delta x_1^{\text{ds}}(\theta) \leq x(\theta)$. Since $q' < 0$, the bracketed term is at least

$$1 - q(x(\theta)) + \frac{\partial W_1}{\partial x_1}(x(\theta), \theta) > 0,$$

where strict positivity follows from $q(x) < 1$ and the exclusive first-order condition. With $x' < 0$, gross issuance strictly decreases on the separating region and is therefore invertible. Define

$$\theta^{\text{ds}}(\Delta x_1) = (\Delta x_1^{\text{ds}})^{-1}(\Delta x_1), \quad b^{\text{ds}}(\Delta x_1) = b(\theta^{\text{ds}}(\Delta x_1)).$$

For each positive prescribed issue, the required share of cash proceeds is

$$\alpha^{\text{ds}}(\Delta x_1) = \frac{b^{\text{ds}}(\Delta x_1)}{q(x_0 + \Delta x_1 - b^{\text{ds}}(\Delta x_1)) \Delta x_1}.$$

The net-proceeds identity and $\tau \geq 0$ ensure that the prepayment does not exceed those proceeds. At zero issuance, the prepayment is zero.

Finally, existing lenders receive the prepayment and retain the remaining claim, so their payoff in state θ is $(x_0 - b(\theta))q(x(\theta)) + b(\theta) = x(\theta)q(x(\theta)) - \tau(\theta)$. Lemma OA.2 therefore gives feasibility of initial financing, and the sweep replicates the exclusive allocation along the prescribed issuance path. $\square$

OA.4.3 Short-Term Debt

As a contrast to these transfer-like clauses, we now introduce short-term debt. Debt maturity is not the paper's central focus, but the exercise is useful because it shows that changing maturity alone does not provide state-contingent compensation to existing lenders.

Specifically, we allow the borrower to finance the upfront investment at $t = 0$ using a combination of short-term debt b that matures at $t = 1$, and long-term debt x_0 that matures at $t = 2$. Short-term debt is due after the discount-factor shock θ is realized. Because the borrower is financially constrained, it must issue new debt to repay the short-term debt b due at that time.

It becomes immediately clear that short-term debt must be riskless because there is no cash-flow risk at $t = 1$.² The next result shows that short-term debt crowds out long-term debt when rollover does not pool all types. Under full pooling, by contrast, maturity is indeterminate.

Proposition OA.2 (Short-Term Debt). *Let $I^p := x_1^*(\theta)q(x_1^*(\theta))$, and let x_1^b be the smaller solution to $xq(x) = I$ (and the unique solution when $I = I^{\max}$). Then:*

1. *If $I < I^p$, the unique optimal maturity structure is $x_0 = 0$ and $b = I$. At $t = 1$, total debt is*

$$x_1(\theta) = \max\{x_1^*(\theta), x_1^b\}.$$

2. *If $I^p \leq I \leq I^{\max}$, optimal total debt is fully pooled at x_1^b . Maturity is indeterminate: every*

$$x_0 \in [0, x_1^b], \quad b = (x_1^b - x_0)q(x_1^b), \quad x_1^L = x_1^b$$

is optimal. In particular, the all-short-term contract $x_0 = 0$ and $b = I$ is optimal, but not unique.

When $I < I^p$, the borrower finances the entire investment with short-term debt and, at $t = 1$, chooses the efficient debt $x_1^*(\theta)$ whenever it raises enough to repay $b = I$ and otherwise borrows x_1^b . Issuing long-term debt creates dilution incentives on the states in which rollover does not bind. Short-term debt avoids this distortion, which makes $x_0 = 0$ uniquely optimal in this regime.

When $I \geq I^p$, the rollover requirement is high enough to pool all types at x_1^b . The same pooled allocation can be financed with any combination of short- and long-term debt in the proposition. Replacing one maturity with the other changes neither

²A proof goes as follows. If the borrower were to default under some report $\hat{\theta}$ (resulting in zero payoff), it would instead choose a report $\hat{\theta}'$ that avoids default and yields a positive payoff. Thus, truthful reporting requires that default never occur, so that short-term debt must be riskless.

initial proceeds nor total debt at $t = 1$, so the maturity composition is not uniquely determined.

These results should be interpreted cautiously because the analysis abstracts away several costs and risks of short-term debt. Firms rarely rely only on short-term debt in practice. For instance, our model ignores transaction costs associated with rolling debt over and assumes that there are no uncertain intermediate cash flows. In a related setting with such uncertainty, [Hu, Varas, and Ying \(2026\)](#) show that borrowers issue long-term debt when short-term debt becomes risky. Outside the fully pooled regime, maturity alone also fails to reproduce the state-contingent compensation delivered by performance-sensitive debt or contingent prepayment.

Proof of Proposition [OA.2](#)

We prove Proposition [OA.2](#) by first showing how optimized surplus varies with long-term debt, then distinguishing partial from full pooling.

A contract (b, x_0, x_1^L) specifies short-term debt due at $t = 1$, long-term debt due at $t = 2$, and a limit on total debt. To repay b at $t = 1$, the borrower must satisfy

$$q(x_1(\theta))(x_1(\theta) - x_0) \geq b.$$

When $x_1^L \in [x_0, x_1^f(\theta, x_0)]$, write $x_1^L = x_1^f(\theta_L)$. Feasibility requires $q(x_1^L)(x_1^L - x_0) \geq b$. Since $\bar{\theta} = 1$ and $x_1^f(\bar{\theta}, x_0) = x_0$ by Lemma [2](#), continuity gives a cutoff $\theta_H \in [\theta_L, \bar{\theta}]$ satisfying

$$q(x_1^f(\theta_H))(x_1^f(\theta_H) - x_0) = b.$$

If the rollover floor exceeds $x_1^f(\theta, x_0)$, all types pool and this cutoff representation does not apply. We treat that case directly below. Within the cutoff range, the borrower solves

$$\begin{aligned}
& \max_{\substack{\theta_L, \theta_H \in \Theta \\ \theta_L \leq \theta_H}} \int_{\underline{\theta}}^{\theta_L} W_1(x_1^f(\theta_L), \theta) dG(\theta) \\
& + \int_{\theta_L}^{\theta_H} W_1(x_1^f(\theta), \theta) dG(\theta) + \int_{\theta_H}^{\bar{\theta}} W_1(x_1^f(\theta_H), \theta) dG(\theta)
\end{aligned} \tag{OA.15}$$

subject to

$$\begin{aligned}
& x_0 \left[\left(q(x_1^f(\theta_L)) - q(x_1^f(\theta_H)) \right) G(\theta_L) + \int_{\theta_L}^{\theta_H} \left(q(x_1^f(\theta)) - q(x_1^f(\theta_H)) \right) dG(\theta) \right] \\
& + x_1^f(\theta_H) q(x_1^f(\theta_H)) \geq I.
\end{aligned}$$

The value of long-term debt. Let $W_0(x_0)$ be the optimized value in (OA.15). Normalize its Lagrangian by $1 + \lambda$, where $\lambda \geq 0$ is the financing multiplier, and set $\delta = 1/(1 + \lambda)$. Write $D(\theta) = \partial x_1^f(\theta, x_0)/\partial x_0 > 0$ and define

$$\begin{aligned}
A_L &:= \int_{\underline{\theta}}^{\theta_L} \left[x_0 q'(x_1^f(\theta_L)) + \delta(\theta_L - \theta) q(x_1^f(\theta_L)) \right] dG(\theta), \\
A_H &:= \int_{\theta_H}^{\bar{\theta}} \left[x_0 q'(x_1^f(\theta_H)) + \delta(\theta_H - \theta) q(x_1^f(\theta_H)) \right] dG(\theta) + (1 - \delta) \theta_H q(x_1^f(\theta_H)).
\end{aligned}$$

For $\delta > 0$, the envelope derivative of the normalized Lagrangian has the same sign as $W'_0(x_0)$ and is

$$\begin{aligned}
\mathcal{D}(x_0) &= x_0 \int_{\theta_L}^{\theta_H} q'(x_1^f(\theta)) D(\theta) dG(\theta) + D(\theta_L) A_L + D(\theta_H) A_H \\
&+ (1 - \delta) \left[\left(q(x_1^f(\theta_L)) - q(x_1^f(\theta_H)) \right) G(\theta_L) \right. \\
&\quad \left. + \int_{\theta_L}^{\theta_H} \left(q(x_1^f(\theta)) - q(x_1^f(\theta_H)) \right) dG(\theta) \right]. \tag{OA.16}
\end{aligned}$$

Suppose first that $\theta_L < \theta_H$. The first-order conditions for the two thresholds give $A_L = A_H = 0$. The same conclusion applies at $\theta_L = \underline{\theta}$, because $G(\underline{\theta}) = 0$. An optimum with $\theta_H = \bar{\theta}$ must have $\delta = 1$, in which case $A_H = 0$ as well; otherwise, a

small decrease in θ_H raises the normalized objective. Hence,

$$\begin{aligned} \mathcal{D}(x_0) = x_0 \int_{\theta_L}^{\theta_H} q'(x_1^f(\theta)) D(\theta) dG(\theta) \\ + (1 - \delta) \left[\left(q(x_1^f(\theta_L)) - q(x_1^f(\theta_H)) \right) G(\theta_L) \right. \\ \left. + \int_{\theta_L}^{\theta_H} \left(q(x_1^f(\theta)) - q(x_1^f(\theta_H)) \right) dG(\theta) \right]. \quad (\text{OA.17}) \end{aligned}$$

Both terms are weakly negative. If $x_0 > 0$, the first term is strictly negative because $q' < 0$, $D > 0$, and $g > 0$.

If $\theta_L = \theta_H$, the ordering constraint binds. At an interior pooled cutoff, adding the two stationarity conditions cancels its multiplier and gives $A_L + A_H = 0$. Since $D(\theta_L) = D(\theta_H)$, the cutoff terms in (OA.16) cancel. The remaining terms vanish because the cutoffs coincide, so

$$\mathcal{D}(x_0) = 0. \quad (\text{OA.18})$$

The pooled contracts below give the same conclusion at $\underline{\theta}$ and $\bar{\theta}$. At a transition between separating and pooled solutions, both one-sided envelope derivatives are nonpositive. Thus W_0 is weakly decreasing on its feasible domain, strictly decreasing along any segment with $x_0 > 0$ and $\theta_L < \theta_H$, and flat only under full pooling.

Partial pooling: $I < I^p$. At $x_0 = 0$, flexible debt is efficient: $x_1^f(\theta, 0) = x_1^*(\theta)$. The borrower sets $b = I$, and rollover gives

$$x_1(\theta) = \max\{x_1^*(\theta), x_1^b\}, \quad x_1^b q(x_1^b) = I,$$

where x_1^b is the smaller root. Since $xq(x)$ increases below its maximizer, $I < I^p$ is equivalent to $x_1^b < x_1^*(\underline{\theta})$. There is therefore a unique $\theta_H \in (\underline{\theta}, \bar{\theta})$ such that $x_1^*(\theta_H) = x_1^b$. The allocation has a separating region of positive measure. Moreover, the multiplier on the financing constraint is positive: relaxing the rollover requirement lowers debt for a positive measure of types whose debt is fixed at x_1^b and strictly raises surplus. Thus $\delta < 1$, and the second term in (OA.17) makes the right derivative at $x_0 = 0$ strictly negative. Since W_0 cannot subsequently increase, no $x_0 > 0$ can attain the same value. It follows that $x_0 = 0$ and $b = I$ are uniquely optimal.

Full pooling: $I^p \leq I < I^{\max}$. Here $x_1^b \geq x_1^*(\theta) \geq x_1^*(\theta)$, so at $x_0 = 0$ rollover pools all types at x_1^b . Every maturity mix

$$x_0 \in [0, x_1^b], \quad b = (x_1^b - x_0)q(x_1^b), \quad x_1^L = x_1^b$$

raises $x_0q(x_1^b) + b = I$ and implements the same allocation. Indeed, for $z \in [x_0, x_1^b]$,

$$\frac{d}{dz} [(z - x_0)q(z)] = q(z) + zq'(z) - x_0q'(z) > 0,$$

because repayment increases below x_1^b and $q' < 0$. Thus debt below x_1^b cannot repay b , while the debt limit rules out higher debt.

To compare with separating contracts, define

$$x_0^p = x_1^b \left[1 - \frac{1 - \theta}{\varepsilon(x_1^b)} \right] \geq 0.$$

The flexible-debt first-order condition gives $x_1^f(\theta, x_0^p) = x_1^b$, so the pooled allocation belongs to the cutoff problem at $x_0 = x_0^p$ and $\theta_L = \theta_H = \underline{\theta}$.

If $x_0 < x_0^p$, then $x_1^f(\theta_H, x_0) \leq x_1^f(\underline{\theta}, x_0) < x_1^b$ for every $\theta_H \in \Theta$. The bracketed term in the financing constraint (OA.15) is nonpositive because $\theta_L \leq \theta_H$ and both x_1^f and q are decreasing. Financing capacity is therefore at most

$$x_1^f(\theta_H, x_0)q(x_1^f(\theta_H, x_0)) < x_1^bq(x_1^b) = I,$$

where the strict inequality uses increasing repayment below the smaller root. Hence the cutoff problem is infeasible for $x_0 < x_0^p$. For $x_0 \geq x_0^p$, the monotonicity of W_0 rules out an improvement over the pooled contract at x_0^p . All the maturity mixes above are therefore optimal. At $I = I^p$, $x_0^p = 0$.

Maximum financing: $I = I^{\max}$. For every feasible contract, rollover implies

$$x_0q(x_1(\theta)) + b \leq x_1(\theta)q(x_1(\theta)) \leq I^{\max}.$$

Raising $I^{\max}$ initially requires both inequalities to bind almost surely. Since $xq(x)$ has the unique maximizer x^m , debt is fully pooled at $x_1^b = x^m$ and rollover binds. Every $x_0 \in [0, x^m]$ with $b = (x^m - x_0)q(x^m)$ and $x_1^L = x^m$ implements this allocation,

as stated in Proposition [OA.2](#).

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