Intermediary Financing without Commitment

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Abstract

Intermediaries reduce agency problems through monitoring, but credible monitoring requires sufficient retention until the loan matures. We study credit markets when intermediaries cannot commit to retention. Two structures are examined: investors lending alongside an all-equity bank and investors lending through the bank via short-term debt. With a commitment to retention, they are equivalent. Without commitment, the all-equity bank sells loans and reduces monitoring over time. Short-term debt encourages the intermediary to retain loans and incentivizes monitoring. Our analysis explains intermediaries' reliance on short-term debt - it enables the intermediary to internalize monitoring externalities.

Keywords: commitment; durable-goods monopoly; financial intermediaries; monitoring; dynamic games; optimal control in stratified domains;.

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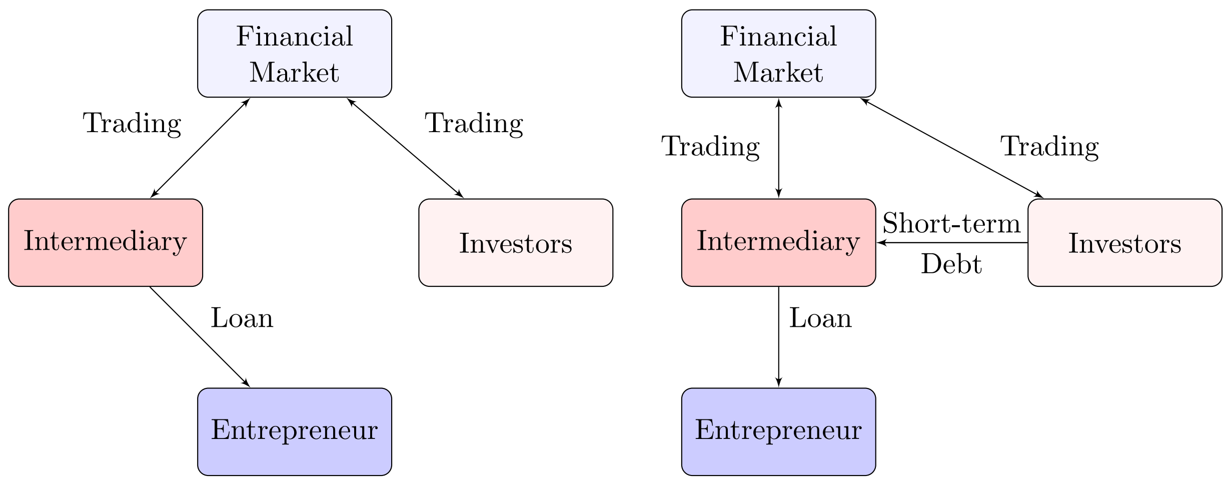
1 Introduction

Financial intermediaries conduct valuable services and, therefore, benefit the real economy. To appropriately align intermediaries' incentives, the optimal financing arrangement typically involves them retaining a fraction of loans as the skin in the game; otherwise, the incentives can be misaligned. With the development of the secondary loan market, intermediaries' commitment to loan retentions is limited (Drucker and Puri, 2009). This paper studies the equilibrium dynamics in loan sales and monitoring when intermediaries *cannot* commit to their retentions. We show that short-term debt can resolve the bank's lack of commitment problem and, therefore, align the incentives for monitoring and loan sales.

We build on the classic model of Holmstrom and Tirole (1997) in which banks can monitor to increase an entrepreneur's borrowing capacity. To maintain incentives for monitoring, the bank is required to retain a sufficient fraction of loans on its balance sheet. We assume the bank has no commitment to hold the loan, but rather has the option to sell loans. Given this, we study two types of financial intermediaries: a bank with all equity financing (E-bank) and a bank with access to short-term debt markets (S-bank). The first regime aligns with the certification structure, while the second corresponds to the intermediation structure as described Holmstrom and Tirole (1997). As shown in Figure 1, in the all-equity bank regime, the bank lends its own funding alongside other investors. In the case of a short-term debt bank, the bank borrows short-term debt from investors and then lends a collection of its own funds and those from investors. The E-bank does not internalize the positive effect that its monitoring has on other investors. Therefore, the lack of commitment induces the bank to sell its loans gradually, and the bank's monitoring intensity declines over time. By contrast, the S-bank does not sell ex-post, because selling would immediately increase the interest rate charged on short-term debt. Hence, the short-term debt leads the bank to internalize the full value of its monitoring.

More specifically, we model an entrepreneur endowed with an investment opportunity, which requires a fixed-size of investment and pays off some final cash flows at a random time in the future. She has limited personal wealth and needs to borrow to make up the investment shortfall. Due to moral hazard in effort choices, she can only pledge a fraction of the final output to outside creditors, including banks and investors. Banks have a higher cost of capital, but only they can monitor to reduce the entrepreneur's private benefits. Although monitoring increases the project's pledgeable income and enables the entrepreneur to borrow more, it also entails a physical cost. Therefore, a credible monitor needs to retain a sufficient fraction of loans as its skin in the game.

We depart from Holmstrom and Tirole (1997) by introducing a competitive financial market, in which the bank is allowed to trade its loans and issue short-term debt against the loans. The bank's loan retention will be the state variable in our model. Loans are rationally priced, and therefore,



(a) All-equity bank (E-bank)

(b) Short-term debt bank (S-bank)

Figure 1: All-equity vs. Short-term debt bank

the prices depend on the bank's incentives to monitor both contemporaneously and in the future. If the bank has sold or is expected to sell a large fraction of the loans, it will monitor less, and consequently, the price of loans will fall. This price impact deters the bank from selling the loans too aggressively.

Although loan sales and debt issuance are equivalent in a model under the bank's commitment to its retention, they lead to different dynamics in loan sales and monitoring if there is no such commitment. An E-bank has incentives to sell loans because it faces a higher cost of capital than investors, making it the inefficient loan holder. The price of the loan will drop when the bank tries to sell because investors anticipate reduced retention and monitoring. However, the price impact is insufficient to deter loan sales, because the bank has multiple future selling opportunities and cannot commit to not exploiting them. Since selling a small loan fraction only slightly reduces monitoring, the initial price impact will be minimal, and the bank will be incentivized to sell a positive amount. However, after this initial sale, the bank faces increased incentives to sell more loans. This is because the price impact from diminished monitoring now applies to only the remaining loan fraction retained by the bank, which is smaller. In equilibrium, the bank trades off the immediate trading gains versus the drop in its future payoff, including the drop in the loan's valuation as well as the decline in the expected payments it can collect upon the project maturing. The price of the loan is such that the E-bank does not benefit from its ability to trade loans at all. The driving force is the time-inconsistency problem - the bank cannot credibly commit to maintaining the loan's value through monitoring in the future with the opportunities to sell. Each loan sale makes the bank's remaining stake smaller, reducing its incentive to internalize monitoring externalities. This dynamic leads to a gradual loan sales and less monitoring over time.

In contrast, an S-bank that issues short-term debt does not have incentives to sell its loans. This is because if the bank tries to sell, it would not only depress the loan's price, but more importantly, it would increase the interest rate on the short-term debt that was used to finance the loan, making it more costly for the bank. Since the bank constantly needs to roll over or refinance its short-term debt, an elevated interest rate acts as a punishment against loan sales. Consequently, despite having no explicit contractual commitment to retain loans, the bank finds it optimal to keep the entire loan portfolio on its balance sheet.

The different results across the two regimes relate to a monitoring externality. Monitoring benefits investors, but banks bear the costs, leading to sub-optimal monitoring (the free-rider problem). For the E-bank, this externality, combined with a lack of commitment to retention, causes it to sell loans and reduce monitoring probability over time. However, for an S-bank, the externality is internalized by short-term debt. Short-term debt pricing immediately reflects any reduction in monitoring by the bank attempting to sell loans. This forces the bank to maintain monitoring incentives as both it and investors share the costs and benefits.

Our paper provides an explanation for why intermediaries rely on short-term debt. We show that an S-bank issuing long-term debt behaves like an E-bank, selling off loans over time. Crucially, our mechanism requires that the intermediary's short-term debt is exposed to credit risk and this risk is fairly priced by markets. If, instead, the bank issues riskless short-term debt, or if its risky short-term debt is not fairly priced, then the S-bank exhibits the same loan sale behavior as an E-bank. We also study the model in which the bank issues long-term debt. Results show that such bank behaves similarly to an E-bank and gradually sells its loans over time, leading to a reduction in monitoring intensity. This occurs because long-term debt does not provide the constant repricing mechanism that short-term debt offers.

2 A Two-period Example

Before proceeding to the main model, we present a simplified example illustrating the basic economic tradeoff. The detailed calculations in this example can be found in Appendix A.

2.1 Model Setup

There are three dates, and the model has one entrepreneur, one bank, and a competitive set of investors. All parties are risk-neutral and have limited liability. We assume investors do not discount the future, whereas the bank and the entrepreneur discount the payoff one period ahead by

$\delta < 1$. We analyze two types of banks: one where the bank is entirely financed by equity (referred to as an *E-bank*), and another where the bank can issue short-term debt (referred to as an *S-bank*).

Figure 2 describes the timing. The entrepreneur has a project that requires her effort at $t = 3$, after which the outcome is realized. The project produces R with probability $p_H = 1$ if the entrepreneur works but with probability $p_L = 0$ if the entrepreneur shirks. There are two shirking options: the high option brings private benefits $B > 0$, whereas the low brings $b = 0$.[1] At $t = 3$, before the entrepreneur chooses her effort, the bank can monitor and eliminate the high shirking option by paying a private cost $\tilde{\kappa}$. This cost follows the uniform distribution on $[0, \bar{\kappa}]$ and is realized at the beginning of $t = 3$. For illustrative purposes, we assume that the entrepreneur does not have any skin in the game and has pledged all the cash flows R as a loan to the bank and investors.

The financial market is open at $t = 1$ and $t = 2$, when the bank can trade the loan and the S-bank can additionally issue debt. We normalize the total share of the loan to one. Let θ_0 be the share of the loan retained by the bank at the beginning of $t = 1$ and $1 - \theta_0$ be that retained by investors. In particular, our results hold for $\theta_0 = 1$. After trading, the bank retains a share θ_1 and θ_2 of the loan at the end of $t = 1$ and $t = 2$, respectively. An E-bank does not issue any debt, whereas an S-bank can issue one-period debt D_1 and D_2 at $t = 1$ and $t = 2$ in the financial market.

$t = 1$	$t = 2$	$t = 3$
- Bank sells loan $\theta_0 - \theta_1$	- Bank repays debt D_1	- Monitoring cost $\tilde{\kappa}$ is realized.
- Bank issues debt D_1	- Banks sells loan $\theta_1 - \theta_2$	- Bank decides whether to monitor.
	- Bank issues debt D_2	- The entrepreneur chooses to work or shirk.
		- Cash flows are realized. Payments are settled.

Figure 2: Timing in the two-period model

2.2 All-equity financed bank

We solve the model via backward induction. At $t = 3$, given a realization of the monitoring cost κ, the bank's IC in monitoring and the implied the probability of bank monitoring are

$$\kappa \leq \theta_2 R \Rightarrow p_2(\theta_2) = \frac{\theta_2 R}{\bar{\kappa}},$$

[1] Therefore, the entrepreneur is indifferent between working and the low shirking option. In this case, we assume that she chooses to work. The low shirking option is redundant in this illustrative example but not in the main model with $b \geq 0$.

Under $p_H = 1$, and $p_L = 0$, $p_2(\theta_2)$ is also the probability that the project generates R. At $t = 2$, the E-bank chooses loan sales $\theta_1 - \theta_2$ to maximize its expected payoff, so

$$\Pi_2^E(\theta_1) = \max_{\theta_2 \in [0,1]} \delta \underbrace{\left(p_2(\theta_2)\theta_2 R - \int_0^{\theta_2 R} \frac{\kappa}{\bar{\kappa}} d\kappa \right)}_{\text{loan payment - monitoring cost}} + \underbrace{q_2(\theta_2)(\theta_1 - \theta_2)}_{\text{loan sales}},$$

where $q_2(\theta_2) = p_2(\theta_2)R$ is the price of the loan. It is easily shown that the bank sells loans

$$\theta_2 = \frac{\theta_1}{2-\delta} < \theta_1.$$

At $t = 1$ given any θ_0, the E-bank's expected payoff is

$$\Pi_1^E(\theta_0) = \max_{\theta_1 \in [0,1]} \quad \underbrace{q_1(\theta_1)(\theta_0 - \theta_1)}_{\text{loan sales}} + \underbrace{\delta \Pi_2^E(\theta_1)}_{\text{payoff at } t=2} .$$

When investors buy the loan at $t = 1$, they anticipate the bank to sell loans at $t = 2$, and such anticipation is reflected in the loan price q_1 so that $q_1(\theta_1) = q_2(\theta_2(\theta_1))$. We show the bank sells loans

$$\theta_1 = \frac{\theta_0}{2-\delta} < \theta_0$$

at a price $q_1(\theta_1) = \frac{R^2}{\bar{\kappa}} \frac{\theta_1}{2-\delta}$, and the bank's expected payoff is $\Pi_1^E(\theta_0) = \frac{R^2}{2\bar{\kappa}} \frac{\theta_0^2}{(2-\delta)^2}$.

2.3 Short-term debt financed bank

At $t = 3$, the bank's IC in monitoring and the implied probability of bank monitoring are

$$\kappa \leq \theta_2 R - D_2 \Rightarrow p_2(\theta_2, D_2) = \frac{\theta_2 R - D_2}{\bar{\kappa}}.$$

At $t = 2$, given that investors do not discount the future, $p_2(\theta_2, D_2)$ is also the price they are willing to pay per unit of debt D_2. For a given θ_1, an S-bank chooses $\theta_1 - \theta_2$ and D_2 to maximize

$$\Pi_2^S(\theta_1) = \max_{\theta_2 \in [0,1], D_2 \geq 0} \quad \delta \underbrace{\left(p_2(\theta_2, D_2)(\theta_2 R - D_2) - \int_0^{(\theta_2 R - D_2)} \frac{\kappa}{\bar{\kappa}} d\kappa \right)}_{\text{loan payment - debt repayment - monitoring cost}}$$
$$+ \underbrace{p_2(\theta_2, D_2) D_2}_{\text{debt issuance}} + \underbrace{q_2(\theta_2, D_2)(\theta_1 - \theta_2)}_{\text{loan sales}}$$

$$\text{s.t.} \quad D_2 \leq \theta_2 R.$$

This issuance constraint $D_2 \leq \theta_2 R$ arises because if D_2 exceeds $\theta_2 R$, the bank will always default: it can never pay off the debt at $t = 3$. It is easily derived that the first-order condition on θ_2 and D_2 are linearly dependent so that the choice between θ_2 and D_2 is undetermined.[2] Therefore, for the S-bank, loan sales and debt issuance are equivalent at $t = 2$. This equivalence also implies

$$\Pi_2^S(\theta_1) = \Pi_2^E(\theta_1).$$

At $t = 1$, the S-bank's payoff from loan sales and debt issuance is

$$\begin{aligned}\Pi_1^S(\theta_0) = \max_{\theta_1, D_1} \quad & \underbrace{D_1}_{\text{debt issuance}} + \underbrace{q_1(\theta_1, D_1)(\theta_0 - \theta_1)}_{\text{loan sales}} + \underbrace{\delta\left(\Pi_2^S(\theta_1) - D_1\right)}_{\text{payoff at } t=2} \\ s.t.\ D_1 \leq \Pi_2^S(\theta_1).\end{aligned}$$

Note that the one-period debt D_1 will be paid with certainty at $t = 2$ and is therefore riskless. As in the case of an all-equity bank, investors anticipate that the bank can sell loans or issue debt at $t = 2$, so the loan price is $q_1(\theta_1, D_1) = q_2(\theta_2(\theta_1), D_2(\theta_1))$.[3] We prove in the appendix that

$$\theta_1 = \theta_0, \qquad D_1 = \Pi_2^S(\theta_1).$$

In other words, the S-bank finds it optimal to refrain from selling loans but instead borrow the maximum possible amount of short-term debt. In contrast to the results at $t = 2$, loan sales and debt issuance are no longer equivalent at $t = 1$. The bank prefers debt issuance strictly over loan sales: it does not sell any loan but issues the maximum amount of debt against it. The bank's expected payoff in this case is given by

$$\Pi_1^S(\theta_0) = \frac{R^2}{2\bar{\kappa}} \frac{\theta_0^2}{2 - \delta} > \Pi_1^E(\theta_0).$$

2.4 Results Summary

Due to the lack of commitment, an E-bank has incentives to sell loans at both $t = 1$ and $t = 2$. These loan sales result in diminished monitoring incentives for the E-bank. For an S-bank at $t = 2$, issuing short-term debt and selling loans are perfect substitutes. This equivalence occurs because the financial market will not open again after $t = 2$, eliminating further opportunities for the bank

[2] In particular, as the derivative of the objective function with respect to θ_2 is proportional to the derivative with respect to D_2, there is a continuum of combinations of θ_2 and D_2 achieving the maximum. The bank is indifferent between any pair θ_2 and D_2 that provides the optimal exposure, that is, that satisfies $\frac{\theta_2 R - D_2}{\bar{\kappa}} = \frac{R}{\bar{\kappa}} \frac{\theta_1}{2-\delta}$.

[3] Note that even though the optimal solution $\{\theta_2, D_2\}$ for a given θ_1 is not unique, any solution will generate the same default probability so there is a unique loan price.

to dilute the investors. Consequently, the commitment problem no longer exists. In this case, loan sales and debt issuance become equivalent methods for the bank to reduce its skin in the game, a result that generally holds in static settings.

The $t = 1$ results for an S-bank show that the equivalence breaks down in dynamic settings. This happens because investors at $t = 1$ anticipate that the financial market will open at $t = 2$ so that the commitment problem will emerge. The $t = 1$ loan buyers are concerned with the bank's decisions of loan sales and debt issuance at $t = 2$ and anticipate to be diluted. In contrast, $t = 1$ creditors are not concerned with the bank's $t = 2$ decisions, as they will be fully repaid before the bank sells loans or issues debt again. Therefore, the S-bank strictly prefers debt issuance to loan issuance.

Below, we show that all these results continue to hold in a fully-dynamic model. The comparisons $\Pi_2^S(\theta_1) = \Pi_2^E(\theta_1)$ and $\Pi_1^S(\theta_0) > \Pi_1^E(\theta_0)$ suggest that the payoff difference between an S-bank and an E-bank increases as opportunities for loan sales and debt issuance grow. However, a two-period model is insufficient to fully characterize this comparison, and extending the discrete-time model beyond two periods becomes intractable.[4] Therefore, we develop the fully-dynamic model in continuous time, which allows for a clear characterization of the equilibrium and associated trading dynamics. In addition, the tractability associated with the continuous-time formulation enables us to explore several extensions to the baseline model.

The bank's problem, with multiple opportunities for loan sales and debt issuance, resembles the durable-goods monopoly problem introduced by Coase (1972). In our context, the bank can be viewed as the monopolist, with claims to the entrepreneur's project cash flows representing the durable goods. This literature demonstrates that Coasian forces prevent the monopoly from reaping any gains. Similarly, we show in the continuous-time model that our E-bank derives no benefit from its loan-selling ability. The literature also reveals that a monopolist can overcome the commitment problem by renting rather than selling goods (Bulow, 1982). In our model, short-term debt of the S-bank resemble the leasing solution to the durable goods monopoly problem.

3 The Model

3.1 Agents and Technology

Time is continuous and goes to infinity: $t \in [0, \infty)$. There are three groups of agents: one entrepreneur (she) – the borrower; one bank; and investors. All agents are risk-neutral and have limited liability. The entrepreneur starts out with cash level A, whereas the bank and investors have

[4] Even in this two-period example, we are only able to obtain closed-form solutions in the special case of $p_H = 1$, $p_L = 0$, $b = 0$, and $F(\cdot)$ follows the uniform distribution.

deep pockets. We assume investors do not discount future cash flows, whereas the entrepreneur and the bank discount the future at a rate $\rho > 0$. Investors in our model should be interpreted as institutional investors such as sovereign wealth funds, hedge funds, insurance companies, and cash-rich companies.

At time 0, the entrepreneur has access to a project that requires a fixed investment size $I > A$. Thus, she needs to borrow at least $I - A$. The project matures at a random time τ, which arrives upon a Poisson event with intensity $\phi > 0$. Define $\Phi = \frac{\phi}{\rho+\phi}$ as the effective time discount the entrepreneur and the bank apply to the project's final cash flows. At τ, the project generates the final cash flows R in the case of success and 0 in the case of failure. The probability of success is p_H if the entrepreneur works at τ, and $p_L = p_H - \Delta$ if she shirks. Two options of shirking are available: the high option brings private benefit B, which exceeds b, the private benefit of the low option. We assume the project's expected payoff is always higher if the entrepreneur works; that is, $p_H R > p_L R + B$.

3.2 Monitoring, Financial Structures, and Contracts

At $t = 0$, the entrepreneur contracts with the bank. Banks in the model should be broadly interpreted as any lender that is capable of costly reducing the agency frictions. At τ, the project matures, and the bank can monitor to eliminate the high shirking option. To do so, it needs to pay private monitoring cost $\tilde{\kappa} > 0$, where $\tilde{\kappa} \in [0, \bar{\kappa}]$ has a distribution with $F(\cdot)$ and $f(\cdot)$ being the cumulative distribution function (CDF) and probability density function (PDF), respectively. The stochastic-cost assumption smooths the bank's equilibrium monitoring decisions, which become a continuous function of its loan retention. Stochastic costs can be interpreted as variations in legal and enforcement costs, or simply fluctuations in the costs of hiring loan officers.[5] Figure 3 describes the timing.

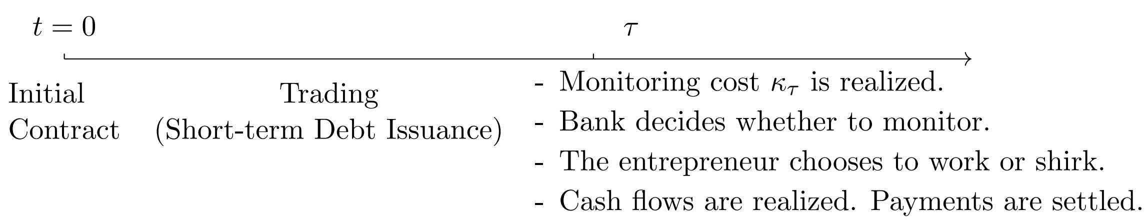


Figure 3: Timing

We study two financial regimes. In the first regime, the bank is all equity financed (henceforward

[5] Two alternative formulations will generate results equivalent to that of a stochastic monitoring cost: The first is to introduce a continuous distribution of private benefits, and the second one is to assume that monitoring is only effective with some probability. This probability is assumed to vary continuously with the bank's monitoring effort.

referred to as an E-bank) and puts its own funds in the entrepreneur's venture to certify it will monitor, which then attracts investors to invest in the venture as well. One example of this type of bank is the lead investment bank in loan syndication. Under limited liability, no agent receives anything if the project fails. If the project succeeds, let R_f be the cash flows retained by the entrepreneur, and $R_o = R - R_f$ be the scheduled payments to outside creditors, namely, the bank and investors. For the remainder of this paper, we also refer to R_o as *loans*.

In the second regime, investors do not directly invest in the entrepreneur's project. Instead, they lend to the bank, which then lends to the entrepreneur a collection of its own funds and the money from investors.[6] One example of this type of bank is a shadow bank, which borrows short-term debt from lenders. The bank offers short-term debt contracts $\{D_t, y_t\}$ to investors over time (henceforward referred to as an S-bank), where D_t is the amount of debt and y_t is the associated interest rate. We model short-term debt as one with instant maturity. Debt with instant maturity is the continuous time analogous to one-period debt in discrete time. The only difference between an S-bank and an E-bank is that the former can issue short-term debt dynamically.

Given that the entrepreneur has (weakly) the highest cost of capital among all the agents, she should retain as little stake as possible. Therefore, in both regimes, it is optimal to let the entrepreneur retain $R_f = b/\Delta$, which guarantees she will work if the bank monitors. Therefore, $R_o = R - b/\Delta$.[7]

3.3 Trading and Pricing in the Financial Market

A competitive financial market is open after time 0 in which loans can be traded.[8] We normalize the total share of loans outstanding to one and use θ_t to denote the bank's retention at time t. In our model, θ_t will be the payoff-relevant state variable in the Markov perfect equilibrium introduced later. Sometimes θ_t is also referred to as the bank's skin in the game, which is publicly observable. Before trading starts, the bank's initial retention is $\theta_0 = 1$ in both regimes. We consider trading strategies that admit both smooth and atomistic trading, as well as mixed strategies over the time of atomistic trades. A Markov trading strategy is defined as $(\theta_t)_{t\geq 0}$ being a Markov process.

[6] The money from investors should not be interpreted as FDIC-insured deposits. Instead, it comes from investors who are sensitive to new information regarding the bank's underlying business. These investors put their money in the bank in the form of short-term debt. In practice, one can interpret short-term debt as brokered deposits, repurchase agreements, wholesale lending, and commercial papers. More broadly, they can be interpreted as "money-market preferred stock," which carries a floating dividend rate that is reset periodically to maintain the stock's market value at par. The distinction between equity and debt is unimportant in our setup when cash flows are binary, with one realization being zero. As we show below, the crucial feature is that current information about the bank promptly becomes impounded into the rate it pays to capital suppliers.

[7] The entrepreneur can also borrow directly from investors, but the borrowing capacity is (weakly) lower.

[8] We assume the entrepreneur's retention R_f is not tradable, or equivalently, the entrepreneur can commit to holding onto R_f on the balance sheet.

The price of loans depends on whether the entrepreneur works or shirks, which in turn depends on the probability of bank monitoring and, therefore, its net exposure. Let $(\theta_t, D_t) = (\theta, D)$. Conditional on the project maturing, let $p(\theta, D)$ be the equilibrium probability of success; investors of the loan receive

$$d(\theta, D) = p(\theta, D) R_o \tag{1}$$

per share. Let $q(\theta, D)$ be the price of the loan per share. In a competitive financial market, the price is given by the expected present value:

$$q(\theta, D) = \mathbb{E}\left[d(\theta_\tau, D_\tau) \Big| \theta_t = \theta, D_t = D\right], \tag{2}$$

where the expectation operator is taken with respect to the equilibrium path of $\{\theta_s, D_s\}_{t \le s \le \tau}$.

The probability of success $p(\theta, D)$ differs in the two regimes. Let κ be the realization of the stochastic monitoring cost $\tilde{\kappa}$ at τ. An E-bank has $D = 0$, and with retention θ, it chooses to monitor if and only if

$$p_H \theta R_o - \kappa \ge p_L \theta R_o \Rightarrow \kappa \le \kappa_E \coloneqq \Delta R_o \theta, \tag{3}$$

where $\Delta \coloneqq p_H - p_L$. An S-bank with retention θ and short-term debt D monitors if and only if

$$p_H(\theta R_o - D) - \kappa \ge p_L(\theta R_o - D) \Rightarrow \kappa \le \kappa_S \coloneqq \Delta(R_o \theta - D). \tag{4}$$

From now on, we use subscripts E and S to differentiate the two regimes.

Assumption 1 restricts the (expected) monitoring cost to be sufficiently low.

Assumption 1.

$$\int_0^{\Delta R - b} \kappa dF(\kappa) \le \Phi F(\Delta R - b)(\Delta R - b) - (1 - \Phi) p_L (\Delta R - b).$$

This assumption leads to the following result. If the bank always retains the entire loan (i.e., $\theta_t \equiv 1,\ \forall t \le \tau$), the bank's payoff exceeds that if it immediately sells the entire loan and never monitors. If this assumption is violated, the expected monitoring cost will be too high, so the bank never monitors in equilibrium.

We assume that – for all retention levels θ – the S-bank's payoff (defined later) is globally concave in the amount of short-term debt issuance. This assumption is stronger than necessary but simplifies the analysis by allowing us to rely on the first-order condition to characterize the short-term debt issuance policy.[9] It holds for various probability distributions, including the uniform distribution and the truncated exponential distribution where the rate parameter is bounded above by $\frac{1}{\bar{\kappa}}\log(1+\Phi\Delta)$.[10]

Assumption 2. *The distribution function $f(\tilde{\kappa})$ satisfies the following for $\forall\kappa\in[0,\bar{\kappa}]$*

1. $-\Phi\left[(p_L+F(\kappa)\Delta)\,f(\kappa)+f^2(\kappa)\,\Delta^2 D\right]<\Delta f'(\kappa)\,D<f(\kappa)\,,\forall D\in[0,p_H R_o\theta],\ \forall\theta\in[0,1]\,.$
2. $\frac{f'}{f^2}<\frac{\Phi}{1-\Phi}\Delta.$

4 Equilibrium

Subsection 4.1 examines the equilibrium of an equity-funded bank (E-bank). Our analysis shows that due to a commitment problem, the E-bank gradually sells its loans, resulting in diminished monitoring incentives. This exercise serves as a useful benchmark. In addition, as we will show in subsection 5.3, a bank relying on long-term debt behaves similarly to an E-bank. Recent literature suggests that deposits, despite their demandable nature, share more similarities with long-term debt (Drechsler et al., 2021). In light of the recent regional banking crisis, the FDIC has proposed to impose a long-term debt requirement for certain insured depository institutions.[11] Our findings indicate that such a reform could exacerbate the commitment problem and further reduce the bank's monitoring incentives. Furthermore, there are episodes where banks' access to short-term debt markets are significantly impaired, with the most recent example being the financial crisis (Gorton et al., 2020). Our analysis of the all-equity financed bank scenario can provide insights into bank behavior during such periods of stress when short-term funding becomes distressed. Subsection 4.2 explores the equilibrium of a bank funded by short-term debt (S-bank). We demonstrate that the ability to issue short-term debt serves as a disciplining mechanism, mitigating the bank's commitment problem.

[9] All the results go through as long as the first-order condition (together with the respective complementary slackness condition) is necessary and sufficient for the optimality of the short-term debt issuance policy. When this is not case, the optimal short-term debt issuance policy may be discontinuous as a function of retention θ.

[10] Note that when $f'<0$, the only condition that we need to check is $-\Phi\left[(p_L+F(\kappa_S)\Delta)\,f(\kappa)+f^2(\kappa)\,\Delta^2 D\right]<\Delta f'(\kappa)\,D$, which becomes $\Phi\left[\left(p_L+\frac{1-e^{-\beta\kappa}}{1-e^{-\beta\bar{\kappa}}}\Delta\right)+\frac{\beta e^{-\beta\kappa}}{1-e^{-\beta\bar{\kappa}}}\Delta^2 D\right]>\Delta\beta D$. A sufficient condition is that $\Phi p_L>\Delta D\beta\left[1-\Phi\Delta\frac{e^{-\beta\kappa}}{1-e^{-\beta\bar{\kappa}}}\right]$, which is necessarily satisfied if $1<\Phi\Delta\frac{e^{-\beta\kappa}}{1-e^{-\beta\bar{\kappa}}}$. It suffices to check the condition at $\kappa=\bar{\kappa}$, which requires that $\beta<\frac{1}{\bar{\kappa}}\log(1+\Phi\Delta)\,.$

[11] https://www.fdic.gov/news/financial-institution-letters/2023/fil23045.html

4.1 E-Bank Equilibrium

Given that $D \equiv 0$, we write all relevant valuation and prices as functions of θ. If the project matures at time t, the bank receives loan payments net of the monitoring cost

$$\pi_E(\theta) = p_E(\theta) R_o \theta - \int_0^{\kappa_E} \kappa dF(\kappa), \tag{5}$$

where the project succeeds with probability

$$p_E(\theta) := p_L + F(\kappa_E)\Delta, \tag{6}$$

upon which the bank receives $R_o\theta$. Following the Envelope Theorem, it is easily shown that

$$\pi_E'(\theta) = p_E(\theta) R_o + \underbrace{\frac{\partial p_E(\theta)}{\partial \theta} R_o \theta - \kappa_E f(\kappa_E) \frac{\partial \kappa_E}{\partial \theta}}_{=0} = d_E(\theta). \tag{7}$$

Let $G(\theta)$ be the bank's cumulative trading gains. In the case of atomistic trading, the bank's holding jumps from θ to θ^+ and the associated trading gain is $dG(\theta) = q(\theta^+)(\theta - \theta^+)$. Note that trading is settled at a price $q(\theta^+)$ to reflect the price impact. In the case of continuous trading, $dG(\theta) = -q(\theta)\dot{\theta}dt$. The bank maximizes the sum of its payoff upon the project's maturation $e^{-\rho(\tau - t)}\pi_E(\theta_\tau)$ and the cumulative trading gains $\int_0^\tau e^{-\rho(s-t)} dG(\theta_s)$. Because τ follows the exponential distribution, the bank's problem can be equivalently written as

$$\max_{\{\theta_t\}_{t \geq 0}} \mathbb{E}\left[\int_0^\infty e^{-(\rho+\phi)t}\Big(\phi \pi_E(\theta_t) dt + dG(\theta_t)\Big)\right], \tag{8}$$

where the expectation operator also allows for mixed strategies in $\{\theta_t\}$. Let V_E be the entrepreneur's expected payoff:

$$V_E = \mathbb{E}\left[\int_0^\infty e^{-(\rho+\phi)t}\phi\left\{\mathbb{1}_{\{\kappa \leq \kappa_E\}} p_H R_f + \mathbb{1}_{\{\kappa > \kappa_E\}}(p_L R_f + B)\right\} dt\right]. \tag{9}$$

If the realized monitoring cost is lower than the threshold κ_E defined in (3), the bank monitors, and the entrepreneur receives $p_H R_f$ in expectation. Otherwise, the bank chooses not to monitor, and the entrepreneur receives the expected return $p_L R_f$ together with the private benefits B.

We consider a Markov perfect equilibrium in which the state variable is the bank's retention θ, henceforth, the E-bank equilibrium.[12]

[12] If $\{\theta_t\}$ is not restricted to the class of Markov processes, one may construct equilibria that are close to the

Definition 1. *An **E-bank equilibrium** is a Markov perfect equilibrium consisting of a price function* $q_E : [0,1] \to \mathbb{R}_+$ *and a trading strategy* $(\theta_t)_{t\geq 0}$ *that satisfy the following:*

1. *Given* $\theta_0 \in [0,1]$*,* $(\theta_t)_{t\geq 0}$ *is a Markov trading strategy that maximizes* (8)*.*

2. *For all* $\theta \in [0,1]$*, the price* $q_E(\theta)$ *satisfies the break-even condition* (2)*.*

In general, the bank can trade loans smoothly or atomistically. We show both types of trading can occur in equilibrium. Let $\Pi_E(\theta)$ be the bank's value function with retention θ.[13] In the smooth-trading region, $\Pi_E(\theta)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho \Pi_E(\theta) = \max_{\dot{\theta}} \ \phi \Big[\pi_E(\theta) - \Pi_E(\theta) \Big] + \dot{\theta} \Big[\Pi'_E(\theta) - q_E(\theta) \Big]. \tag{10}$$

Whereas the left-hand side stands for the bank's required return, the first term on the right-hand side represents the event of the project maturing, in which case the bank receives $\pi_E(\theta)$ defined in (5). The second term captures the overall benefit of trading, which includes the change to the bank's continuation value and the trading gain. A necessary condition for smooth trading is

$$\Pi'_E(\theta) = q_E(\theta), \tag{11}$$

so that the bank is indifferent between trading or not: the per-share trading gain $q_E(\theta)$ is offset by the drop in the bank's continuation value $\Pi'_E(\theta)$. Substituting the indifference condition (11) into (10), we get that in the region of smooth trading,

$$\Pi_E(\theta) = \frac{\phi}{\rho + \phi} \pi_E(\theta) = \Phi \pi_E(\theta). \tag{12}$$

Given that $\Phi \pi_E(\theta)$ is also the bank's continuation value if it never sells any loan until the project matures, (12) implies that in equilibrium, the bank does not benefit from its ability to trade these loans in the financial market at all. The observation that lack of commitment fully offsets the trading gains has been noted in previous models on bargaining (Fuchs and Skrzypacz, 2010; Daley and Green, 2020) and in other corporate finance settings (DeMarzo and Urošević (2006) in trading by a large shareholder, and DeMarzo and He (2021) in leverage dynamics).

commitment solution. In the context of a durable-goods monopoly, Ausubel and Deneckere (1989) show that in the no-gap case, non-Markov equilibria exist in which the seller can achieve payoffs close to the commitment solution. The logic behind the construction is similar to the one in the folk theorem for repeated games. The no-gap case corresponds to the version of our model under $p_L = 0$. At $\theta = 0$, the marginal valuation of investors coincides with the bank's under $p_L = 0$.

[13] An E-bank does not issue debt, so there is no distinction between the bank value and the equity value.

Even though the bank's equilibrium payoff is identical to if it always retained the entire loan, this does not imply the bank will never trade loans on the equilibrium path. In fact, the price of loans if the bank followed a strict no-trade strategy would be too high, incentivizing the bank to sell. The bank sells loans for two reasons: First, due to its higher cost of capital, the bank's marginal valuation is below investors', making it the inefficient holder of the loan. Selling a small loan fraction only slightly reduces monitoring incentives while the initial price impact is minimal, incentivizing some sales. Second, after the initial sale, the bank is willing to sell more because the price impact only accrues to its remaining retention stake, which gets smaller. We characterize the bank's equilibrium trading strategy, which comes from the determination of the equilibrium loan prices. Because investors do not discount future cash flows, $q_E(\theta)$ must satisfy the investors' zero-profit equation whenever the bank trades smoothly:

$$0 = \phi \Big[d_E(\theta) - q_E(\theta) \Big] + \dot{\theta} q_E'(\theta) \,. \tag{13}$$

The left-hand side is the investors' required rate return, which is zero. On the right-hand side, $\phi \Big[d_E(\theta) - q_E(\theta) \Big]$ stands for the event of project maturing, in which case investors receive $d_E(\theta)$ per share, whereas the last term is the change in the loan price due to trading.[14] Combining (11), (12), and (13), and using the relation $d_E(\theta) = \pi_E'(\theta)$, one can derive the following equilibrium trading strategies:

$$\dot{\theta} = -\phi \frac{(1 - \Phi)\, \pi_E'(\theta)}{\Phi \pi_E''(\theta)} < 0. \tag{14}$$

In the smooth-trading region, the bank sells loans over time and its retention declines continuously. Intuitively, the equilibrium loan price is forward-looking and therefore takes into account the bank's monitoring decisions in the future. To satisfy the bank's indifference condition, the equilibrium price of the loan cannot be too high. The only trading strategy consistent with this price requires the bank to sell its loans over time.

So far, we have only considered the case of smooth trading. Meanwhile, the bank also has the option to sell an atom of loans. In general, the bank can sell either a fraction or all the remaining loans. Lemma 4 in the appendix proves the bank will never sell a fraction. This result follows the intuition in standard Coasian dynamic models. Atomistic trading arises whenever the bank has strict incentives to sell. If so, it prefers to sell as fast as possible. Given this result, we are left to check when the bank decides to sell off all the remaining loans at a price $q_E(0)$, where $q_E(0) = p_L R_o$ is the per-share loan price without monitoring. We show that a unique θ_* exists

[14] The terms without subscripts c have been defined in equations (1) and (2).

below which the bank finds it optimal to sell off all the remaining loans; that is, $\theta_* q_E(0) = \Phi \pi_E(\theta_*)$, and $\theta q_E(0) > \Phi \pi_E(\theta)$ if and only if $\theta < \theta_*$.

Next, we derive the trading strategy at $\theta = \theta_*$. On the one hand, the bank cannot hold onto the remaining loans forever because the resulting loan price will be too high to induce the bank to sell. On the other hand, the bank cannot sell smoothly either; if so, shortly afterward, θ will drop below θ_*, and the bank will have strict incentives to sell off the rest of the loan. Furthermore, it cannot be that the bank sells off all the remaining loan immediately after θ_t reaches θ_*, because if so, the price of the loan will experience a deterministic downward jump, inconsistent with equation (2). The only (stationary) trading strategy at θ_* consistent with (2) is for the bank to adopt a mixed strategy:[15] the bank sells off all its remaining loans at time ν, which arrives upon a Poisson event at intensity λ that satisfies

$$q_E(\theta_*) = \mathbb{E}\left[d_E(\theta_\tau)|\theta_t = \theta_*\right] = \frac{\lambda}{\phi+\lambda} d_E(0) + \frac{\phi}{\phi+\lambda} d_E(\theta_*). \tag{15}$$

Proposition 1 summarizes the previous discussion and describes the equilibrium outcome. The formal proof requires verification that the bank's trading strategy is optimal, which is supplemented in the appendix using results from the theory of optimal control in stratified domains.[16]

Proposition 1 (E-bank equilibrium)**.** *A unique **E-bank equilibrium** exists. With $\theta_0 = 1$, the bank sells its loans smoothly at a rate given by equation (14) until $T_* \coloneqq \inf\{t > 0 : \theta_t = \theta_*\}$, after which it sells off its remaining loans at some Poisson rate λ that satisfies (15). The equilibrium loan price is*

$$q_E(\theta) = \begin{cases} \Phi\Big(p_L + F\left(\Delta R_o \theta\right)\Delta\Big) R_o & \theta > \theta_* \\ \Big(p_L + \frac{\phi}{\lambda+\phi} F\left(\Delta R_o \theta_*\right)\Delta\Big) R_o & \theta = \theta_* \\ p_L R_o & \theta < \theta_*. \end{cases} \tag{16}$$

If $\Pi_E(1) < I - A$, the entrepreneur cannot borrow enough to invest at $t = 0$.

[15] The delay can also be deterministic, but the equilibrium is no longer within the class of a Markov perfect equilibrium. The equilibrium would also depend on the time since the bank's retention reached θ_*. The price q_t would not be stationary, and it would depend on the trading history before time t.

[16] Due to the discontinuity in the price function $q_E(\theta)$, the HJB equation (10) is discontinuous at θ_*. This technical problem can be sidestepped using (discontinuous) viscosity solution methods.

4.2 S-Bank Equilibrium

Let $D_t = D$ be the outstanding debt of an S-bank at time t. If the project matures, the bank's equity holders receive loan payments net of debt repayments and monitoring cost:

$$\hat{\pi}_S(\theta, D) = \hat{p}_S(\theta, D)(\theta R_o - D) - \int_0^{\kappa_S} \kappa dF(\kappa), \tag{17}$$

where the project succeeds with probability

$$\hat{p}_S(\theta, D) \coloneqq p_L + F(\kappa_S)\Delta, \tag{18}$$

upon which the bank's equity holder receives $\max\{\theta R_o - D, 0\}$. Besides trading gains, an S-bank also receives income from issuing short-term debt. In particular, the bank's net income from debt issuance at time t is $dD_t - y_t D_t dt$, where

$$y_t = \hat{y}(\theta, D) = \phi(1 - \hat{p}_S(\theta, D)) \tag{19}$$

compensates the default risk borne by creditors.[17] The bank trades loans and issues short-term debt to maximize the expected payoff upon the project maturing, together with the cash flows from short-term debt issuance and trading gains $dG(\theta_t)$; that is,

$$E(\theta, D) = \max_{\{\theta_t,\, D_t\}} \mathbb{E}\left[\int_0^{\infty} e^{-(\rho+\phi)t}\Big(\phi\hat{\pi}_S(\theta_t, D_t)\,dt + [dD_t - \hat{y}(\theta_t, D_t)\,D_t dt] + dG(\theta_t)\Big)\right]. \tag{20}$$

The choice of D_t in (20) is restricted by the bank's limited liability, which imposes an issuance constraint as illustrated below in (21). Lemma 1 shows that we can solve short-term debt issuance and loan trading separately.

Lemma 1. *The maximization problem* (20) *is equivalent to solving*

$$\phi\pi_S(\theta) \coloneqq \max_{D \le \Pi_S(\theta)} \left\{\phi\left[\hat{p}_S(\theta, D)\theta R_o - \int_0^{\kappa_S} \kappa dF(\kappa)\right] + \rho D\right\}, \tag{21}$$

[17] A heuristic derivation of (19) goes as follows. The approximate probability that the project matures over a period of length dt is ϕdt, and the default probability is $1 - \hat{p}_S(\theta_t, D_t)$ within this period. Given there is zero recovery upon default, the promised payoff to creditors $1 + y_t dt$ needs to satisfy the break-even condition

$$1 = (1 - \phi dt)(1 + ydt) + \hat{p}_S\phi dt(1 + ydt) + (1 - \hat{p}_S)\phi dt \times 0.$$

The expression for y_t follows by ignoring higher-order terms.

where

$$\Pi_S(\theta) = E(\theta, D) + D = \max_{(\theta_t)_{t\geq 0}} \int_0^{\infty} \mathbb{E}\left[e^{-(\rho+\phi)t}\Big(\phi\pi_S(\theta_t)dt + dG(\theta_t)\Big)|\theta_0 = \theta, D_0 = D\right]. \quad (22)$$

Given this result, we can suppress the problem's dependence on D_t and use θ_t as the state variable. Note that in (21), debt issuance is bounded by the endogenous constraint $D \leq \Pi_S(\theta)$, which arises from the bank's limited liability.[18] In (22), the left-hand side $\Pi_S(\theta)$ includes $E(\theta, D)$, the value to the bank's equity holders, and D, the value of short-term debt. One implication of (22) is that even though the bank's equity holders decide its trading strategy, maximizing the bank's equity value is equivalent to maximizing the total bank value, because debt D is continuously repriced.

We use V_S to denote the entrepreneur's expected payoff:

$$V_S = \mathbb{E}\left[\int_0^{\infty} e^{-(\rho+\phi)t}\phi\left\{\mathbb{1}_{\{\kappa\leq\kappa_S\}}p_H R_f + \mathbb{1}_{\{\kappa>\kappa_S\}}(p_L R_f + B)\right\}dt\right].$$

The expression differs from (9) in that the threshold cost for monitoring is replaced by κ_S. We look for a Markov perfect equilibrium in state variable θ_t, henceforth, the S-bank equilibrium.

Definition 2. *An* ***S-Bank Equilibrium*** *is a Markov perfect equilibrium consisting of a price function $q_S\colon [0,1] \to \mathbb{R}_+$, a trading strategy $(\theta_t)_{t\geq 0}$, a debt-issuance policy $D^*\colon [0,1] \to \mathbf{R}_+$, and the interest-rate function $y\colon [0,1] \to \mathbf{R}_+$ that satisfy the following:*

1. *For all $\theta \in [0,1]$, the debt-issuance policy $D^*(\theta)$ solves (21).*
2. *Given $\theta_0 = 1$, $(\theta_t)_{t\geq 0}$ is Markov trading strategy that maximizes (22).*
3. *For all $\theta \in [0,1]$, the price $q_S(\theta)$ satisfies the break-even condition (2).*
4. *For all $\theta \in [0,1]$, the interest rate $y(\theta) := \hat{y}(\theta, D^*(\theta))$ satisfies (19).*

The analysis of the S-bank equilibrium has two steps: debt issuance and loan trading.

4.2.1 Short-term Debt Issuance

Dividing both sides of (21) by $\rho + \phi$, we can define

$$\mathcal{V}(D,\theta) := \Phi\pi_S(\theta) = \Phi\left[\hat{p}_S(\theta, D)\,\theta R_o - \int_0^{\kappa_S} \kappa dF(\kappa)\right] + (1-\Phi)\,D. \quad (23)$$

[18] Here, $\Pi_S(\theta)$ is the bank's value function given its retention θ, which implicitly assumes debt issuance has been chosen at the optimal level. Therefore, this constraint involves a fixed point for the value function $\Pi_S(\theta)$. Hu et al. (2021) use the same technique to reduce the problem's dimensions. A similar problem is analyzed in Abel (2018).

The term in the bracket is the net payoff to the bank and its creditors: with probability $\hat{p}_S(\theta, D)$, the project succeeds so that they receive θR_o; $\int_0^{\kappa_S} \kappa dF(\kappa)$ is the expected monitoring cost. The last term in (23) is the value from issuing debt. An increase in D reduces the bank's monitoring incentive and therefore reduces the first term. Meanwhile, an increase in D also reduces the bank's funding cost and therefore increases the last term. The optimal D balances the two effects. Under Assumption 2, $\mathcal{V}(D, \theta)$ is concave in D. Meanwhile, the bank's equity holders' limited liability constraint requires that for any θ, $D \leq \Pi_S(\theta)$. We have the following result.

Lemma 2. *Let $D^*(\theta)$ be the optimal choice of short-term debt. There exists a threshold θ_D where $D^*(\theta) = \Pi_S(\theta)$ if $\theta \leq \theta_D$. If $\theta > \theta_D$, then $D^*(\theta) < \Pi_S(\theta)$, and the bank finances the remaining loan $\Pi_S(\theta) - D^*(\theta)$ using its own capital.*

According to Lemma 2, the bank finances the loan using both short-term debt and bank capital when θ is relatively high. When θ is relatively low, the loan is only financed via short-term debt. This result implies that higher levels of bank capital are associated with more retention and monitoring.

4.2.2 Trading

Next, we turn to the maximization problem (22) and study how an S-bank trades its loans over time. Following similar steps in the E-bank equilibrium, the term $\dot{\theta}\left(\Pi'_S(\theta) - q_S(\theta)\right)$ must vanish in the smooth-trading region, so the bank's continuation value satisfies the HJB:

$$\rho \Pi_S(\theta) = \phi \left[\pi_S(\theta) - \Pi_S(\theta) \right], \tag{24}$$

and the equilibrium price is determined by the indifference condition $\Pi'_S(\theta) = q_S(\theta)$. The trading strategy follows from the investors' zero-profit condition that is similar to (13):

$$0 = \phi \left[d_S(\theta) - q_S(\theta) \right] + \dot{\theta} q'_S(\theta) \Longrightarrow \dot{\theta} = -\phi \frac{d_S(\theta) - q_S(\theta)}{q'_S(\theta)}. \tag{25}$$

Applying the Envelope Theorem in Milgrom and Segal (2002), we get

$$\pi'_S(\theta) = d_S(\theta) + \left[f(\kappa_S) \frac{\partial \kappa_S}{\partial \theta} \Delta \right] D^*(\theta) + z(\theta) \Pi'_S(\theta), \tag{26}$$

where $z(\theta)$ is the Lagrange multiplier of the debt-issuance constraint $D \leq \Pi_S(\theta)$.[19] The equilibrium trading rate becomes

$$\dot{\theta} = \phi \frac{(1-\Phi)(1-p(\theta))R_o + \Phi z(\theta)\left(\Pi'_S(\theta) - R_o\right)}{\Phi \pi''_S(\theta)}. \tag{27}$$

Lemma 3. *For any θ, $\dot{\theta} > 0$ holds under the optimal short-term debt issuance policy $D(\theta) = D^*(\theta)$.*

To complete the characterization of the equilibrium, we need to consider the case in which the bank trades an atom of loans. Similar to the E-bank equilibrium, we can show a unique $\theta_\dagger$ exists such that the bank sells off all the remaining loans at a price $q(0) = p_L R_o$ if $\theta < \theta_\dagger$. However, when $\theta = \theta_\dagger$, unlike in the E-bank equilibrium, the mixed strategy is no longer needed. Instead, the bank buys the loan smoothly so that θ will increase toward $\theta = 1$. Therefore, the price of the loan satisfies $q_S(\theta_\dagger) = \Phi\pi'_S(\theta_\dagger)$.[20] The following proposition describes the results.

Proposition 2 (S-bank equilibrium)**.** *There is a unique **S-bank equilibrium**:*

- *For all $\theta \in [0,1)$, there exists a unique optimal debt issuance policy $D^*(\theta)$, where $D^*(\theta) < \Pi_S(\theta)$ if and only if $\theta > \theta_D$.*
- *For all $\theta \in [0,1]$, the interest rate is $y(\theta) = \phi\left(1 - p_L - F\left(\Delta\left(R_o\theta - D^*(\theta)\right)\right)\Delta\right)$.*
- *For $\theta \in [\theta_\dagger, 1)$, the bank buys loans following (27) until θ_t reaches one. For $\theta \in (0, \theta_\dagger)$, the bank immediately sells all the remaining loans.*
- *The equilibrium loan price is*

$$q_S(\theta) = \begin{cases} \Phi\pi'_S(\theta) & \theta \geq \theta_\dagger \\ p_L R_o & \theta < \theta_\dagger, \end{cases} \tag{28}$$

  where $\pi'_S(\theta)$ is defined in (26).

The bank initially lends $\Pi_S(1)$. On the equilibrium path, the bank retains $\theta = 1$ until the project matures, and issues short-term debt $D = D^*(1)$.

Proposition 2 shows that an S-bank finds it optimal to retain the loan and issues short-term debt against it. In other words, the bank prefers issuing short-term debt against the loan over

[19]The expression for $z(\theta)$ is available in equation (46) of the appendix.

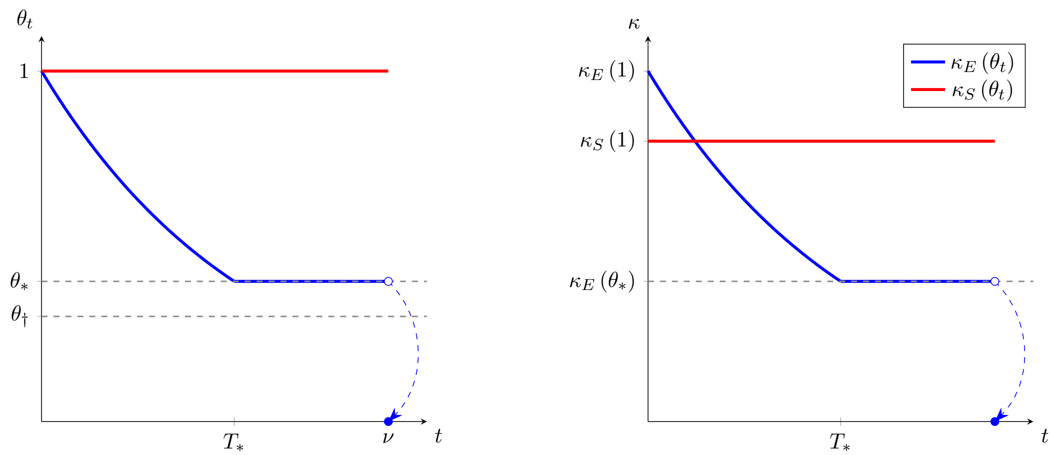
[20]In both regimes, the price function $q(\theta)$ is discontinuous. Whereas in the E-bank equilibrium, the bank trades toward the discontinuity point (i.e., $\dot{\theta}(\theta_*+) < 0$), in the S-bank equilibrium, the bank trades away from the discontinuity point (i.e., $\dot{\theta}(\theta_\dagger+) > 0$). The construction of the equilibrium (and the analysis of the bank's optimal control problem) is simpler in this latter case because the trajectory of θ_t does not "see" the discontinuity.

directly selling it. The reason is that investors would increase the interest rate on the short-term debt had the bank reduced its retention.

5 Equilibria Comparison, Mechanism, and Discussion

5.1 Comparing Retention and Monitoring

Figure 4 compares the bank's loan retention and the implied monitoring intensity after a loan is originated. In both panels, the blue lines describe the E-bank equilibrium, whereas the red ones describe the S-bank equilibrium. In the E-bank equilibrium, the bank first sells the loan smoothly. After θ_t reaches θ_*, the bank sells off all the remaining loans at time ν, following a stochastic delay. By contrast, an S-bank starts with $\theta_0 = 1$ and always retains the loan until the project matures.



(a) Retention (b) Monitoring

Figure 4: Retention and Monitoring Dynamics

The different patterns in retention also imply different probabilities of monitoring. Specifically, a comparison between the threshold monitoring costs κ_E in (3) and κ_S in (4) shows that under the same level of retention θ, an S-bank monitors less due to its outstanding debt D. Meanwhile, an S-bank could have higher retention θ, so the overall comparison of monitoring is ambiguous. The right panel of Figure 4 plots the monitoring threshold $\kappa_E(\theta_t)$ and $\kappa_S(\theta_t)$. A higher threshold is associated with a higher probability of monitoring if the project matures at time t. Early on, κ_E is higher, but eventually, κ_E falls below κ_S. Our model thus predicts that a E-bank conducts more monitoring if the loan matures early, whereas an S-bank conducts more monitoring if the loan matures late.

To understand the different retention patterns between the two structures, we compare the marginal value of retention to the bank's payoff in the smooth trading region. Given $\Pi_E(\theta) = \Phi\pi_E(\theta)$ and $\Pi_S(\theta) = \Phi\pi_S(\theta)$, we directly compare $\pi'_E(\theta)$ with $\pi'_S(\theta)$. In the E-bank equilibrium, (7) shows $\pi'_E(\theta) = d_E(\theta)$, so the marginal value equals expected cash flows per share of the loan. However, $\Phi\pi'_E(\theta) = \Phi d_E(\theta)$ falls below investors' valuation (without trade) $d_E(\theta)$, due to discount rate differences, making the bank an inefficient loan holder and creating gains from trade. While all creditors benefit from monitoring, the bank alone bears the cost. This monitoring externality implies when selling, the bank does not internalize the reduced the value of others' loan stakes, leading to suboptimal monitoring over time under an E-bank.

In the S-bank equilibrium, (26) shows an increase in θ leads to two additional benefits. The first benefit is characterized by the term $\left[f\left(\kappa_S\right)\frac{\partial\kappa_S}{\partial\theta}\Delta\right]D^*$, which shows the marginal effect of retention θ on $f\left(\kappa_S\right)\frac{\partial\kappa_S}{\partial\theta}\Delta$, the incremental probability that debt will be repaid. As a result, more retention enables the bank to issue *cheaper* debt. The second benefit is captured by the last term $z(\theta)\Pi'_S(\theta)$, where $z(\theta) > 0$ if $D^*\left(\theta\right) = \Pi_S(\theta)$. Here, an increase in θ relaxes the constraint on debt issuance so that the bank can issue *more* debt. Note that Lemma 3 continues to hold even if this second benefit disappears, that is, even if $z(\theta) = 0$ so that the debt-issuance constraint is slack. The two additional benefits associated with an increase in θ lead to the different result in the S-bank equilibrium when compared to the E-bank equilibrium. The bank wants to increase its retention because more retention enables an S-bank to issue cheaper and more short-term debt. The role of short-term debt is to help the bank internalize the monitoring externality. Indeed, the interest rate of short-term debt reflects the probability of monitoring so that the bank and its creditors share both the benefits and costs of monitoring. In other words, short-term debt creates a market for the services offered by the bank, that is, monitoring, to be fairly priced and constantly repriced.

5.2 A Linear-Quadratic Example

This subsection presents a linear-quadratic example that allows us to obtain closed-form solutions for the initial payoffs. In particular, we specialize the analysis to the case in which the probability of success is $p_L = 0$ if the entrepreneur shirks and the monitoring cost $\tilde{\kappa}$ follows the uniform distribution on $[0, \bar{\kappa}]$. In addition, we assume $\bar{\kappa} \geq \Phi\Delta^2 R_o$, which guarantees the bank never monitors with probability one. Under these parameters, the bank will never sell off all its loans atomistically (that is, $\theta_* = 0$ and $T_* \to \infty$ in the E-bank equilibrium), because the resulting price will be zero. This example is solved following steps in Section 4. We only describe the main results here, leaving the details to the appendix.

With an E-bank, the amount that the entrepreneur can borrow at $t = 0$ and her payoff from the project maturing are

$$L_E = \Pi_E(1) = \frac{\Phi}{2\bar{\kappa}} (\Delta R_o)^2$$
$$V_E(1) = \Phi B - \frac{\Phi}{2-\Phi} \frac{\Delta R_o}{\bar{\kappa}} (B-b).$$

With an S-bank, the uniform distribution leads to a result that the optimal amount of short-term debt without the issuance constraint $D \leq \Pi_S(\theta)$ is a constant $\frac{1-\Phi}{\Phi}\frac{\bar{\kappa}}{\Delta^2}$. The optimal debt issuance is

$$D^*(\theta) = \min\left\{\frac{1-\Phi}{\Phi}\frac{\bar{\kappa}}{\Delta^2}, \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o\theta)^2} - \frac{\bar{\kappa}}{\Delta^2}\right\}.$$

The constraint binds at $\theta = 1$ if and only if $\Phi < \sqrt{\frac{(\bar{\kappa}/\Delta^2)^2}{R_o^2 + (\bar{\kappa}/\Delta^2)^2}}$. The amount that the entrepreneur is able to borrow at $t = 0$ and her payoff from the project maturing are

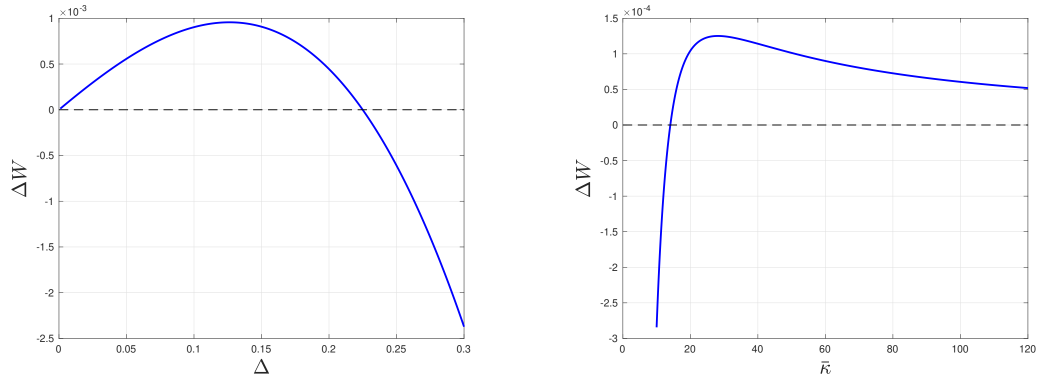
$$L_S = \Pi_S(1) = \Phi\frac{\Delta^2\left[R_o^2 - (D^*(1))^2\right]}{2\bar{\kappa}} + (1-\Phi)D^*(1)$$
$$V_S(1) = \Phi B - \Phi\frac{\Delta(R_o - D^*(1))}{\bar{\kappa}} (B-b).$$

Initial Surplus

We proceed to compare the initial surplus in the two equilibrium regimes.

Proposition 3. *The entrepreneur is always able to borrow more with an S-bank, i.e., $L_S > L_E$. Define $\Delta W = (\Pi_E(1) + V_E(1)) - (\Pi_S(1) + V_S(1))$. Under uniform distribution and $p_L = 0$, if $\Phi < \sqrt{\frac{(\bar{\kappa}/\Delta^2)^2}{R_o^2 + (\bar{\kappa}/\Delta^2)^2}}$ so that the debt issuance constraint is slack at $\theta = 1$, then $\Delta W < 0$, and the S-bank equilibrium has a higher surplus.*

Proposition 3 shows that the borrowing capacity is always higher in the S-bank regime. Moreover, if the debt issuance constraint is slack, the S-bank equilibrium always generates a higher initial surplus for the bank and the entrepreneur. However, if this constraint binds, the S-bank's ability to issue deposits is limited, and the E-bank equilibrium may potentially generate a higher initial surplus. Figure 5 considers some numerical examples to illustrate this possibility. The left panel reveals that when Δ gets sufficiently low, the surplus in E-bank equilibrium is higher, whereas the right panel shows similar patterns as $\bar{\kappa}$ gets sufficiently high. A lower Δ implies reduced benefits from monitoring, while a higher $\bar{\kappa}$ corresponds to a higher cost of monitoring.



(a) ΔW and Δ

(b) ΔW and $\bar{\kappa}$

Figure 5: Initial Surplus

5.3 Long-term Debt

Our analysis has thus far highlighted the crucial feature of short-term debt: that the interest rate can adjust as soon as the bank changes its retention to reflect the credit risks borne by creditors. By doing so, short-term debt resolves the free-riding problem in bank monitoring. This subsection further explores the mechanism by studying long-term debt.

Perpetual Debt Financed Bank

Drechsler et al. (2021) argue that for traditional banks, even though deposits are nominally short-term, they effectively constitute long-term financing. We solve the equilibrium in which the bank can issue only long-term debt $D_0 = D$ at $t = 0$ (henceforward referred to as a P-bank), that is, debt that only matures with the project at τ. After $t = 0$, the bank is no longer allowed to issue any further debt. The bank's incentive compatibility constraint continues to be characterized by (4), captured by $\kappa_P \coloneqq \Delta\left(R_o\theta - D\right)$ The definition of $\hat{p}_P\left(\theta, D\right)$ is the same as (18). Similarly, $\hat{\pi}_P\left(\theta, D\right)$ is defined as in (17). However, note that in these equations, D is now chosen at time $t = 0$. Given so, we suppress these functions' dependence on D, and therefore refer to $\kappa_P\left(\theta\right)$, $p_P\left(\theta\right)$, and $\pi_P\left(\theta\right)$, respectively. After $t = 0$, the bank only chooses its trading strategy and solves

$$\max_{\{\theta_t\}_{t\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-(\rho+\phi)t}\left(\phi\pi_P\left(\theta_t\right)dt + dG\left(\theta_t\right)\right)\right],$$

which is isomorphic to the E-bank's problem in (8). Therefore, the solution method, as well as the equilibrium outcomes, follow the E-bank equilibrium in subsection 4.1. The HJB equation (10) and the zero-profit equation (13) continue to hold, implying that after $t = 0$, the bank sells its loans and reduces its retention over time. Under the parameter conditions presented in the linear-quadratic example, we get the following payoff and pricing functions (details in appendix):

$$V_P(D) = \frac{\phi}{\rho + \phi} B - (B - b) \frac{\Delta \phi}{\bar{\kappa}} \frac{R_o - D}{2\rho + \phi} \tag{29}$$

$$\Pi_P(D) = \frac{\Phi \Delta^2}{2\bar{\kappa}} (R_o - D)^2 \tag{30}$$

$$q_D(D) = \frac{\Phi \Delta^2}{\bar{\kappa}} (R_o - D), \tag{31}$$

where subscripts P stand for perpetual debt, $V_P(D)$ is the entrepreneur's payoff, $\Pi_P(D)$ the P-bank's payoff, and $q_D(D)$ the price of debt at $t = 0$. The optimal long-term debt D^* is chosen to maximize the initial surplus

$$\max_D \; W_P(D) = V_P(D) + \Pi_P(D) + D q_D(D),$$

with the optimal solution being $D_P^* = \frac{B-b}{\Delta(2-\Phi)}$.

We have examined the use of long-term debt in the absence of lender dispersion (i.e., $\theta_0 = 1$). When considering the combination of long-term debt D and θ_0 that maximizes initial surplus, we find that the solution is indeterminate. This happens because loans and perpetual debt have similar maturity characteristics, resulting in only the optimal net exposure being determined. This outcome is reminiscent of the equivalence observed in the two-period example. Given the earlier result that the S-bank prefers issuing short-term debt to selling loans, this equivalence also implies that short-term debt is preferable to long-term debt.

Adjustment of Long-Term Debt

The previous analysis considers the case where long-term debt is only issued at time zero. One could consider the situation when banks issue and adjust long-term debt over time dynamically. Also, instead of considering perpetual debt, we could follow the literature on debt rollover and consider exponentially maturing long-term debt. However, such an extension would require a significant detour from our current analysis, greatly increasing the model's complexity due to the need to keep track of two state variables: the bank's retention and the amount of outstanding debt.[21]

That being said, some recent developments in the literature may offer insights into the potential

[21] Given that monitoring incurs a fixed cost, our setting is not homogeneous of degree one, so we cannot scale by assets to reduce the problem's dimension to a single variable.

equilibrium dynamics of such a model. In our current setting, if the bank could issue both instantly maturing short-term debt and long-term debt, it would likely opt to issue short-term debt exclusively. The reason is that long-term debt suffers from a similar commitment problem as loan sales suffer in our setting. In Hu et al. (2021), we have considered a related model to study dynamic capital structure. There we show that in the absence of a hedging benefit, the equity holders of a firm would exclusively rely on short-term debt rather than issue long-term debt.

Generally speaking, in the absence of commitment, long-term debt is not expected to impose a significant disciplinary effect on the borrower because the constant repricing has a smaller impact (only a fraction of the debt is rolled over), and can also be diluted by new debt issuance. Once long-term debt has been issued, banks have incentives to dilute the value held by long-term creditors. This can be achieved either by selling loans or by issuing additional long-term debt. A key finding in this literature is that, given the lack of commitment, borrowers in equilibrium achieve the same payoff as if they had never issued long-term debt, rendering debt maturity essentially irrelevant (see DeMarzo and He (2021), for example).

6 Extensions and Empirical Relevance

6.1 Extensions

Lender Dispersion

The benchmark model assumes the bank is the only lender. We now show that our results in both regimes extend to multiple lenders. Specifically, an E-bank gradually sells its loans over time, whereas an S-bank never trades. The exercises here relate to Bolton and Scharfstein (1996), who highlight one benefit of having dispersed lenders. They show that borrowing from multiple creditors can discipline a firm's manager by reducing the rents to renegotiation after a strategic default. In our model, the bank has a commitment problem that reduces the ex-post incentives in monitoring, but lender dispersion is not the key reason that resolves the commitment problem.

First, let us consider the model in which the E-bank is one of many lenders. This corresponds to the model in subsection 4.1 with θ_0 being optimally chosen. For $\forall\theta_0 \in (\theta_*, 1]$, the equilibrium dynamics in loan sales, retention, monitoring, and pricing stay qualitatively unchanged: the E-bank starts with θ_0, sells the loan gradually until θ_t reaches θ_*, after which the bank sells the remaining loans at a Poisson rate λ. With a competitive set of lenders, the entrepreneur chooses θ_0 at $t = 0$ to maximize $V_E(\theta_0) + L_E(\theta_0)$, subject to the borrowing constraint

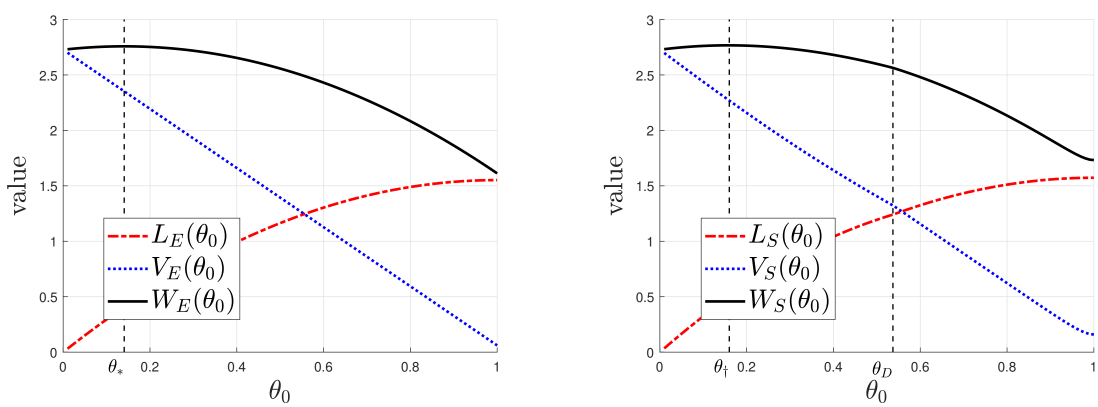
$$L_E(\theta_0) = \underbrace{\Pi_E(\theta_0)}_{\text{bank lending}} + \underbrace{(1-\theta_0)q_E(\theta_0)}_{\text{investors' lending}} \geq I - A.$$

We show in the appendix that $L_E(\theta_0)$ increases monotonically so that $\theta_0 = 1$ maximizes the entrepreneur's initial borrowing. However, as shown by the solid line of Figure 6, the total surplus $V_E(\theta_0) + L_E(\theta_0)$ reaches its maximum at an interior point θ_0^*. In the linear-quadratic example, we get

$$L_E(\theta_0) = \Pi_E(\theta_0) + q_E(\theta_0)(1-\theta_0) = \frac{\Phi}{2\bar{\kappa}}(\Delta R_o \theta_0)^2 + \frac{\Phi}{\bar{\kappa}}(\Delta R_o)^2 \theta_0 (1-\theta_0)$$
$$V_E(\theta_0) = \Phi B - \frac{\Phi}{2-\Phi}\frac{\Delta R_o \theta_0}{\bar{\kappa}}(B-b)$$

The optimal solution is interior if $\frac{1}{2-\Phi}\frac{B-b}{\Delta R_o} < 1$, in which case $\theta_0^* = 1 - \frac{1}{2-\Phi}\frac{B-b}{\Delta R_o}$.

Second, let us consider the model in which the S-bank can borrow from multiple lenders. This corresponds to the model in subsection 4.2, but $\theta_0 \in [0,1]$ is optimally chosen by the entrepreneur. The equilibrium is numerically solved (details in appendix). As shown by the dashed line of Figure 6, the optimal solution can also be interior $\theta_0 \in (0,1)$. For $\theta_0 \in (\theta_\dagger, 1]$, as shown in Lemma 3, the S-bank will buy loans over time until θ_t reaches one, after which it retains the loan until the project matures.



(a) Payoffs in E-bank Equilibrium

(b) Payoffs in S-bank Equilibrium

Figure 6: Initial Surplus under Lender Dispersion

Short-Term Debt Pricing

In our model, short-term debt is fairly priced. In practice, however, short-term bank debt benefits from explicit or implicit guarantees (e.g., FDIC insurance) and is not fairly priced (Drechsler et al., 2021). We consider an extension where the bank only partially passes through interest rates to their main short-term debt holders due to these guarantees. One can think of these guarantees as either deposit insurance or the implicit guarantee from a government bailout.[22] We show that once the guarantee becomes sufficiently large, an S-bank no longer retains but instead sells loans over time, just as an E-bank. This exercise reveals that for short-term debt to help the bank overcome the commitment problem, a crucial feature is its fair pricing by the market.

Specifically, we assume the bank only needs to pay a fraction ξ of the interest rate so that equation (19) becomes

$$y_t = \phi\xi\left(1 - \hat{p}_S\left(\theta, D\right)\right),$$

where $\xi \in (0,1)$. Following the analysis in subsection 4.2, we define $\mathcal{V}$, the bank's objective function without trading gains as

$$\mathcal{V}(D,\theta) := \Phi\left[\hat{p}_S\left(\theta, D\right)\theta R_o - \int_0^{\kappa_S} \kappa dF\left(\kappa\right)\right] + \left(1-\Phi\right)D + \Phi\left(1 - \hat{p}_S\left(\theta, D\right)\right)\left(1-\xi\right)D,$$

where the new term $\Phi\left(1 - \hat{p}_S\left(\theta, D\right)\right)\left(1-\xi\right)D$ stands for the benefit of the government subsidy. The optimal short-term debt issuance can be solved by taking the first-order condition of $\mathcal{V}$ with respect to D. As in Lemma 3, the issuance constraint is slack when θ is sufficiently high. In this case, the HJB equation implies

$$\dot{\theta} = \phi\frac{R_o\xi\frac{(1-\Phi)-\Phi(1-p(\theta))(1-\xi)}{1-(1-p(\theta))(1-\xi)} - \left(1-\Phi\right)p(\theta)R_o}{\Phi\pi_S''\left(\theta\right)}.$$

When $\xi = 1$, we get the results in subsection 4.2 that $\dot{\theta} = \phi\frac{R_o(1-\Phi)(1-p(\theta))}{\Phi\pi_S''(\theta)} > 0$. Moreover, when $\xi = 0$ so that the entire interest rate is subsidized by the government, $\dot{\theta} = \phi\frac{-(1-\Phi)p(\theta)R_o}{\Phi\pi_S''(\theta)} < 0$, implying the bank sells loans over time. We have the following.

Proposition 4. *A $\xi_\dagger$ exists such that in the S-bank equilibrium, $\dot{\theta} < 0$ if $\xi < \xi_\dagger$ for θ sufficiently large.*

Continuous Monitoring Technology

In our model, the monitoring decision is only made when the project matures. A more realistic model would require the monitoring decision to be made continuously. We consider such an extension and show that our main result–that short-term debt helps the bank overcome the commitment

[22] In the U.S., deposit insurance takes the form of a maximum guaranteed amount that has been \$250,000 since 2010. There is a one-to-one mapping between the maximum insurance amount and the parameter ξ introduced later on. To see this, note one can think about the interest rate as $y_t = 0$ for deposits below $\$250,000$ but following (19) for deposits above the limit. Our parameter ξ captures the fraction of deposits that are above the limit.

problem–remains valid in this setup.[23]

Suppose the project still matures at a rate of ϕ. At maturity, it produces a payoff of R with a probability of 1 unless it fails before that maturity event occurs. The failure probability depends on the bank's monitoring effort: before maturity, the project fails with an arrival rate $\eta(1-m_t)$, where m_t is the bank's monitoring effort chosen at time t. The cost of monitoring is given by a convex function $h(m_t) = \frac{1}{2}km_t^2$. Note that in this model, monitoring directly affects the arrival rate of the cash flows.

For the E-bank, the HJB equation is

$$(\rho+\eta)\,\Pi_E(\theta) = \max_{\dot{\theta},m}\ \phi\Big[R_o\theta - \Pi_E(\theta)\Big] + \eta m \Pi_E(\theta) - h(m) + \dot{\theta}\Big[\Pi_E'(\theta) - q_E(\theta)\Big],$$

which leads to the first-order conditions $\Pi_E'(\theta) = q_E(\theta)$ and $h'(m) = \eta\Pi_E(\theta)$. The loan price is

$$\eta(1-m(\theta))q_E(\theta) = \phi\Big[R_o - q_E(\theta)\Big] + \dot{\theta}q_E'(\theta)\,,$$

where $m(\theta) = (h')^{-1}[\eta\Pi_E(\theta)]$. These two equations have similar interpretations as (10) and (13) in subsection 4.1. Following similar steps there, we get

$$\dot{\theta} = -\rho\frac{q_E(\theta)}{q_E'(\theta)} = -\rho\frac{\Pi_E'(\theta)}{\Pi_E''(\theta)} < 0.$$

For the S-bank, we have the following problem for short-term debt issuance:

$$\phi\pi_S(\theta) := \max_{D\le\Pi_S(\theta)}\{\phi\theta R_o + \rho D\}$$

The constraint is $D \le \Pi_S(\theta)$ is always binding so we get $\phi\pi_S(\theta) = \phi\theta R_o + \rho\Pi_S(\theta)$. The bank's continuation value satisfies the HJB:

$$(\rho+\eta)\Pi_S(\theta) = \max_{\dot{\theta},m}\ \phi\Big[\pi_S(\theta) - \Pi_S(\theta)\Big] + \eta m\Pi_S(\theta) - h(m) + \dot{\theta}\left[\Pi_S'(\theta) - q_S(\theta)\right]$$

where the first order conditions are $\Pi_S'(\theta) = q_S(\theta)$ and $h'(\theta) = \eta\Pi_S(\theta)$. From the loan pricing equation,

$$\eta(1-m(\theta))q_S(\theta) = \phi\left(R_o - q_S(\theta)\right) + \dot{\theta}q_S(\theta),$$

[23] We are grateful to an anonymous referee for this suggestion.

we plug in $q_S(\theta) = \Pi'_S(\theta)$ and $q'_S(\theta) = \Pi''_S(\theta)$ to get that

$$\dot{\theta} = 0.$$

Clearly, the main results in our main model carry over to this extension. Specifically, an E-bank will gradually sell its loans over time, whereas an S-bank never trades. The reason, again, is due to the role of short-term debt, which gets constantly repriced. Detailed analysis is supplemented in Appendix E.3.

S-bank with Loss-Absorbing Equity and Riskless Debt

We now extend the model by introducing loss-absorbing equity to the S-bank. This allows the S-bank to issue riskless short-term debt. Our analysis reveals that when constrained to issue only riskless short-term debt, the S-bank behaves similarly to the E-bank, gradually selling loans over time. However, given the choice, the S-bank always optimally selects a level of short-term debt that is exposed to credit risks, rather than opting for riskless debt. Under optimal short-term debt, the S-bank overcomes the commitment and never sells the loan. These results demonstrate that short-term debt can effectively resolve the commitment problem only when it is exposed to credit risks.

We assume if the project fails and generates nothing, the bank is able to pay the creditors up to $X \in (0, D)$. One can think about X as the level of the bank's risk-absorbing equity or the liquidity required to put aside in case of bank failure. Therefore, the bank's incentive compatibility constraint in monitoring becomes

$$p_H(\theta R_o - D) + (1 - p_H)(-X) - \kappa \geq p_L(\theta R_o - D) + (1 - p_L)(-X) \Rightarrow \kappa \leq \kappa_S := \Delta(R_o\theta + X - D).$$

The definition for $\hat{p}_S(\theta, D)$ stays unchanged from (18), whereas $\hat{\pi}_S(\theta, D)$ becomes

$$\hat{\pi}_S(\theta, D) = \hat{p}_S(\theta, D)(\theta R_o - D) + (1 - \hat{p}_S(\theta, D))(-X) - \int_0^{\kappa_S} \kappa dF(\kappa)$$

to include equity holders' losses when the project fails. If the bank is mandated to issue riskless short-term debt so that a constraint $D \leq X$ is imposed, then $\hat{y}(\theta, D) = 0$. In equilibrium, equity holders will always issue up to the limit so that $D^*(\theta) = X$. The rest of the equilibrium follows exactly the E-bank equilibrium.

Without the constraint of $D \leq X$, it is never optimal for the bank to issue riskless debt. In other words, the endogenous choice of debt always satisfies $D > X$, with an interest rate $\hat{y}(\theta, D) = \phi\big(1 - \hat{p}(\theta, D - X)\big)\left(1 - \frac{X}{D}\right)$. To see this, let us define $\tilde{D} \equiv D - X$. Following the same step as in the derivation of equation (23), we can write the bank's payoff function in terms of net debt $\tilde{D} \equiv D - X$:

$$\mathcal{V}(\tilde{D}, \theta) := X + \Phi \left[\hat{p}_S(\theta, \tilde{D})\theta R_o - \int_0^{\kappa_S} \kappa dF(\kappa) \right] + (1 - \Phi)\, \tilde{D}.$$

The rest of the problem becomes similar to the S-bank equilibrium, except that the bank chooses $\tilde{D}$ as opposed to D. Therefore, the equilibrium is similar to that in Proposition 2.

6.2 Related Literature

Short-Term Debt. In our paper, short-term debt resolves the commitment issue through a repricing mechanism whereby the bank's retention choice directly affects the cost (and also the amount) of short-term debt. The role of short-term debt in aligning a bank's incentives has been discussed by the previous literature in banking (Calomiris and Kahn, 1991; Diamond and Rajan, 2001), which emphasizes the demandable feature of debt and the externalities from depositor runs. Calomiris and Kahn (1991) is about stopping a crime in progress through a run, and the prospect of a run creates a reward for information acquisition. In our paper, there is no run and, hence, no need to reward information acquisition. In Diamond and Rajan (2001), there is no crime in progress to be stopped. Instead, uninformed depositors just have to run when being held up — they are solving a severe incentive/rent extraction problem. Our paper is similar in that it also solves the incentive problems by bankers, but different in that repricing with no runs also gives the bank enough incentives to monitor. Our results are also related to Flannery (1994), which emphasizes the timing between investment and debt issuance. By contrast, the mechanism in our model relies crucially on the mismatch of liabilities and assets. The repricing of short-term debt prevents reduced monitoring by the financial intermediary.[24]

Our model's role of short-term debt is closely related to the leasing solution to the durable goods monopoly problem. In particular, a monopolist can overcome the commitment problem by renting the good rather than selling it (Bulow, 1982). Intuitively, the short-term nature of the rental contract does not allow the monopolist to take advantage of early buyers. Instead, any change in rental prices simultaneously affects all buyers, eliminating the monopolist's temptation to discriminate against buyers over time. The associated commitment problem is, therefore, resolved. In our context, an S-bank that issues short-term debt against the project's cash flows can be thought of as a rentor of the claims. The short-term nature of the contract implies that any change in retention is immediately priced by all investors, which eliminates the bank's incentives to sell

[24]We are particularly grateful to Raghuram Rajan for pointing out and interpreting the differences between the three papers discussed in this paragraph.

loans.

Loan Sales and Securitization in Banking. This paper is also related to a large literature on bank monitoring, loan sales, and securitization. Parlour and Plantin (2008) show that the informational advantage acquired via bank screening could lead to illiquidity in the secondary market. Our paper is dynamic, and the source of illiquidity comes from the lack of commitment rather than information asymmetry. The commitment problem is also mitigated when claims are collateralized. Rampini and Viswanathan (2019) emphasize the advantage of intermediaries in collateralizing claims. In their paper, the two regimes studied in our paper are equivalent.

Durable-goods monopoly. The lack of commitment problem was initially recognized by Coase (1972) in the context of durable goods monopoly. This paper belongs to the more recent literature that applies the related insights to corporate finance and banking. The E-bank equilibrium in our model is closely related to DeMarzo and Urošević (2006), who study a large shareholder's tradeoff between monitoring and diversification without commitment. Admati et al. (2018) and DeMarzo and He (2021) study the problem when a borrower cannot commit to its debt level, which leads to the leverage-ratchet effect. A main insight of this literature is that the Coasian force leads to no gains at all, which has also been shown in the context of bargaining by Fuchs and Skrzypacz (2010) and Daley and Green (2020).

6.3 Empirical Relevance

Next, we provide a discussion of the paper's empirical relevance.

Our study is based on the empirical finding that there exists a strong correlation between bank monitoring activities and the bank's retention. Gustafson et al. (2021) use site visits, third-party evaluations, and demands for loan-specific information as proxies for bank monitoring. They show that in syndicated loans, monitoring increases with the lead bank's retention. Moreover, they observe an inverse relationship between bank monitoring intensity and loan interest rates. Focarelli et al. (2008) present evidence of a certification effect: syndicated loans tend to have lower interest rates when a greater portion of the loan facility is held by the arranger. In addition, Wang and Xia (2014) banks tend to exert less effort in post-loan monitoring when they have the option to securitize loans.

Our model predicts that the price of the loan in the secondary market is negatively correlated with banks' skin in the game. Empirically, Irani et al. (2021) collect data on secondary market pricing of loan sales and show that syndicated loans with higher nonbank funding experienced greater downward pressure on secondary market prices during the recent financial crisis.

Our model predicts banks that do not face constant capital market discipline, such as those from short-term creditors, will sell their loans and reduce monitoring. In practice, a bank typically has multiple outstanding loan facilities to a single borrower (Term Loan A, Term Loan B, and revolver). Therefore, one should interpret the one loan in our model as the combination of the bank's credit exposure to a borrower. Our model predicts that an E-bank without any commitment to retention will gradually reduce its overall exposure to the borrower. Meanwhile, S-banks, i.e., those with short-term debt constantly repriced, are subject to discipline. Therefore, the S-bank should not be interpreted as a traditional commercial bank that relies on uninformed and insured retail deposits. Rather, this bank could be interpreted as institutions such as shadow banks, asset-backed commercial paper conduits (ABCP Conduit), or structured investment vehicles (SIV), which rely largely on short-term funding from institutional investors. Relatedly, one should not interpret investors as traditional retail depositors protected by FDIC insurance. Instead, they should be considered as institutional investors, such as non-banks or short-term wholesale funding. Our paper, therefore, emphasizes the role of informed creditors' discipline in bank lending. Irani and Meisenzahl (2017) find that when banks are more exposed to disruptions in the short-term wholesale funding market in the crisis, they have a higher probability of selling loan shares, and this depressed loan prices in the secondary market. Specifically, they show that the result is driven by the bank's exposure to short-term funds, including repos and interbank borrowing.

Our paper provides an explanation for why intermediaries rely on short-term debt. Jiang et al. (2020) document that shadow banks originate long-term loans by issuing short-term debt to a few informed lenders. In our model, banks should be more broadly interpreted as any financial institution capable of lending and monitoring. In that case, our paper can be related to the recent empirical studies that document the role of non-bank lenders and their substitution for bank lending. Specifically, Gopal and Schnabl (2022) study the borrowing decisions by small businesses and document the importance of nonbank lenders such as finance companies. Chernenko et al. (2022) study borrowing decisions by middle-market firms and document the prevalence of nonbank financial intermediaries in this market. To the best of our knowledge, no empirical research has studied the ex-post performance of loans extended by non-bank lenders and how it relates to these lenders' monitoring intensities. Our paper predicts that the performance crucially depends on the liability structure of these non-bank lenders, a hypothesis to be tested in the future.

7 Final Remarks

This paper develops a theory of intermediary financing when banks cannot commit to the retentions on the balance sheet. Our main message is that the intermediary's liability structure impacts the dynamics of lending and monitoring. A bank that finances using long-term claims, such as long-term debt and equity, has incentives to sell loans over time, leading to a gradual reduction in monitoring. By contrast, an S-bank that finances itself by issuing short-term debt does not have incentives to sell loans. As a result, the E-bank lends less than the S-bank. The E-bank has more monitoring during the early periods after loan origination, whereas the S-bank has more monitoring during the later periods.

Throughout the paper, we follow Holmstrom and Tirole (1997) by assuming that all projects financed by an intermediary are perfectly correlated and thus abstract from the bank's ability to pool assets and diversify the risk (Diamond, 1984; DeMarzo, 2005). In the case of many loans, we can interpret the monitoring decision as the bank's investment in its monitoring technology (which might include the adoption of advanced information technology, the hiring of qualified loan officers, more efficient internal governance, etc.). Such investments improve the bank's ability to control bank-specific risk in its portfolio, which cannot be eliminated by diversification.

In our model, informed investors monitor the bank's balance sheet. It is important, though, that investors cannot write binding contracts based on their assessments of the riskiness of the bank. As argued by Flannery (1994), banks specialize in financing non-marketable, informationally intensive assets, and the composition of these assets changes rapidly with new business opportunities. As a result, these assets do not have contractible, easily described risk properties.

Our main focus has been on the bank's ex-post monitoring rather than ex-ante screening (Ramakrishnan and Thakor, 1984). Hu and Varas (2021) shows how zombie lending will emerge in this context when screening takes time, as a relationship bank can signal through either dynamic retention or debt issuance.[25] Given that our focus is on how the bank's liability structure enables commitment to retention, we chose to stay away from these complications introduced by screening.

By focusing on Markov equilibria, we ignore the intermediary's concern for reputation (see Winton and Yerramilli (2021) for some recent work on the role of reputation concerns). While reputation does not directly affect the return to monitoring in our model, it can have an important effect on the dynamics of loan sales. In particular, one can construct an equilibrium in which the commitment problem is mitigated if the intermediary has a long-run reputation (see Ausubel and Deneckere (1989) for a study of the impact of durable good monopolist's reputation concerns and Malenko and Tsoy (2020) in the context of a corporate borrower who cannot commit to its debt level).

[25] See Leland and Pyle (1977) and Ross (1977) for related issues in the static environment.

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Appendix

A Discrete-time Model

We solve a more general version of the model where $p_H, p_L \in [0,1]$, $p_H > p_L$, and $b \geq 0$. For convenience, we define $\Delta \equiv p_H - p_L$ and $R_o = R - b/\Delta$ because the entrepreneur needs to retain $R_f = b/\Delta$. Moreover, we assume that the monitoring cost $\tilde{\kappa}$ follows a distribution $F(\cdot)$ over the interval $[0, \bar{\kappa}]$. The analysis in Section 2 focuses specifically on the case of $p_H = 1$, $p_L = 0$, $b = 0$, and the distribution $F(\cdot)$ is uniform.

The model is solved using backward induction. The model is first solved with general D_2 and D_1. Then, we specialize to the case where $D_2 = D_1 = 0$, corresponding to the E-bank.

At $t = 2$, the bank can either sell the loan $\theta_1 - \theta_2$ or issue debt D_2 against it. The bank's payoff at $t = 2$ is

$$\hat{\Pi}_2(\theta_2, D_2, \theta_1) = \underbrace{\delta\left(p(\theta_2, D_2)(\theta_2 R_o - D_2) - \int_0^{\Delta(\theta_2 R_o - D_2)} \kappa dF(\kappa)\right)}_{\text{bank equity value}} + \underbrace{p(\theta_2, D_2)D_2}_{\text{debt issuance}} + \underbrace{q(\theta_2, D_2)(\theta_1 - \theta_2)}_{\text{trading gains}}.$$

The bank's problem is

$$\begin{aligned}
\Pi_2(\theta_1) = &\max_{\theta_2 \in [0,1], D_2 \geq 0} \hat{\Pi}_2(\theta_2, D_2, \theta_1) \\
&\text{s.t.} \quad D_2 \leq \theta_2 R_o.
\end{aligned}$$

The price of debt and loans is given by

$$\begin{aligned}
p_2(\theta_2, D_2) &= p_L + \Delta F\left(\Delta(\theta_2 R_o - D_2)\right) \\
q_2(\theta_2, D_2) &= p_2(\theta_2, D_2) R_o = \left[p_L + \Delta F\left(\Delta(\theta_2 R_o - D_2)\right)\right] R_o,
\end{aligned}$$

so it follows that $\frac{\partial p_2(\theta_2, D_2)}{\partial \theta_2} = -R_o \frac{\partial p_2(\theta_2, D_2)}{\partial D_2}$. From here, we get that the derivative of the bank's objective function with respect to θ_2 and D_2 are

$$\begin{aligned}
\frac{\partial \hat{\Pi}_2(\theta_2, D_2, \theta_1)}{\partial \theta_2} &= (\delta - 1) p_2(\theta_2, D_2) R_o + \frac{\partial p_2(\theta_2, D_2)}{\partial \theta_2} \left[R_o(\theta_1 - \theta_2) + D_2\right] \\
\frac{\partial \hat{\Pi}_2(\theta_2, D_2, \theta_1)}{\partial D_2} &= (1 - \delta) p_2(\theta_2, D_2) + \frac{\partial p_2(\theta_2, D_2)}{\partial D_2} \left[R_o(\theta_1 - \theta_2) + D_2\right] \\
\Rightarrow \frac{\partial \hat{\Pi}_2(\theta_2, D_2, \theta_1)}{\partial \theta_2} &= -R_o \frac{\partial \hat{\Pi}_2(\theta_2, D_2, \theta_1)}{\partial D_2}.
\end{aligned}$$

By formulating the Lagrangian of the bank's problem, we get

$$\frac{\partial \mathcal{L}_2}{\partial \theta_2} = \frac{\partial \hat{\Pi}_2(\theta_2, D_2, \theta_1)}{\partial \theta_2} + \eta_2 R_o = 0$$
$$\frac{\partial \mathcal{L}_2}{\partial D_2} = \frac{\partial \hat{\Pi}_2(\theta_2, D_2, \theta_1)}{\partial D_2} - \eta_2 = 0,$$

where η_2 is the Lagrange multiplier on the constraint $D_2 \leq \theta_2 R_o$. It follows that $\frac{\partial \mathcal{L}_2}{\partial \theta_2} \propto \frac{\partial \mathcal{L}_2}{\partial D_2}$, so any pair (θ_2, D_2) that solves the first order condition for θ_2 also satisfies the first order condition for D_2. Therefore, the choice of (θ_2, D_2) is undetermined. If we restrict attention $D_2 = 0$, then the first order condition becomes

$$(1-\delta)p_2(\theta_2, D_2)R_o = \frac{\partial p_2(\theta_2, D_2)}{\partial \theta_2} R_o(\theta_1 - \theta_2).$$

Given $\frac{\partial p_2(\theta_2, D_2)}{\partial \theta_2} > 0$, $\theta_2 < \theta_1$. In general, there is no closed-form solution for θ_2 as a function of θ_1. If $b = 0$ and $\kappa \sim U[0, \bar{\kappa}]$ we get that the price of the loan given θ_2 is

$$q_2(\theta_2, 0) = \theta_2 \frac{R^2}{\bar{\kappa}},$$

so the first-order condition becomes

$$\begin{aligned} \frac{\partial \hat{\Pi}_2(\theta_2, 0, \theta_1)}{\partial \theta_2} &= -(1-\delta)q(\theta_2, 0) + \frac{\partial q(\theta_2, 0)}{\partial \theta_2}(\theta_1 - \theta_2) \\ &= -(1-\delta)\theta_2 \frac{R^2}{\bar{\kappa}} + \frac{R^2}{\bar{\kappa}}(\theta_1 - \theta_2) = 0. \end{aligned}$$

From here, we get

$$\theta_2 = \frac{\theta_1}{2-\delta}.$$

Substituting in the bank's objective function, we get

$$\Pi_2^E(\theta_1) = \frac{R^2}{2\bar{\kappa}(2-\delta)}\theta_1^2.$$

Notice that because the solution for the S-bank is undetermined, we have that $\Pi_2^S(\theta_1) = \Pi_2^E(\theta_1)$

First Trading Period

Now we move back and consider the bank financing at $t = 1$. Given any θ_0, the bank's payoff from loan trading and issuing one-period debt at $t = 1$ is

$$\hat{\Pi}_1(\theta_1, D_1, \theta_0) = D_1 + q_1(\theta_1, D_1)(\theta_0 - \theta_1) + \delta\left(\Pi_2(\theta_1) - D_1\right).$$

The bank's problem at $t = 1$ is

$$\Pi_1(\theta_0) = \max_{\theta_1 \in [0,1], D_1 \geq 0} \hat{\Pi}_1(\theta_1, D_1, \theta_0)$$
$$\text{s.t.} \quad D_1 \leq \Pi_2(\theta_1).$$

All Equity-financed Bank We set $D_1 = 0$ so $q_1(\theta_1, D_1) = q_1(\theta_1, 0)$. The first-order condition for θ_1 is

$$\begin{aligned}
\frac{\partial \hat{\Pi}_1(\theta_1, 0, \theta_0)}{\partial \theta_1} &= -q_1(\theta_1, 0) + \frac{\partial q_1(\theta_1, 0)}{\partial \theta_1}(\theta_0 - \theta_1) + \delta \frac{\partial \Pi_2^E(\theta_1)}{\partial \theta_1} \\
&= -q_1(\theta_1, 0) + \frac{\partial q_1(\theta_1, 0)}{\partial \theta_1}(\theta_0 - \theta_1) + \delta q_2(\theta_2(\theta_1), 0) \\
&= -q_1(\theta_1, 0) + \frac{\partial q_1(\theta_1, 0)}{\partial \theta_1}(\theta_0 - \theta_1) + \delta q_1(\theta_1, 0) \\
&= -(1 - \delta) q_1(\theta_1, D_1) + \frac{\partial q_1(\theta_1, 0)}{\partial \theta_1}(\theta_0 - \theta_1),
\end{aligned}$$

where the second line follows the envelope condition. Hence, the first-order condition for θ_1 can be written as

$$\frac{\partial q_1(\theta_1, 0)}{\partial \theta_1}(\theta_0 - \theta_1) = (1 - \delta) q_1(\theta_1, 0) > 0,$$

where

$$\frac{\partial q_1(\theta_1, 0)}{\partial \theta_1} = \frac{\partial q_2}{\partial \theta_2} \frac{\partial \theta_2}{\partial \theta_1} = (R_o \Delta)^2 f\left(\Delta(\theta_2 R_o)\right) \frac{\partial \theta_2}{\partial \theta_1}.$$

It remains to derive the sign of $\frac{\partial \theta_2}{\partial \theta_1}$. Noticing that

$$\frac{\partial \hat{\Pi}_2(\theta_2, 0, \theta_1)}{\partial \theta_2} = (\delta - 1) p_2(\theta_2, 0) R_o + \frac{\partial p_2(\theta_2, 0)}{\partial \theta_2} R_o (\theta_1 - \theta_2),$$

we get

$$\frac{\partial^2 \hat{\Pi}_2(\theta_2, 0, \theta_1)}{\partial \theta_2 \partial \theta_1} = \frac{\partial p_2(\theta_2, 0)}{\partial \theta_2} R_o > 0,$$

which means that $\hat{\Pi}_2(\theta_2, 0, \theta_1)$ satisfies the single crossing property in (θ_1, θ_2). It follows from the monotone comparative results in Milgrom and Shannon (1994) that $\frac{\partial \theta_2}{\partial \theta_1} \geq 0$. We can find a closed-form solution if $b = 0$ and $\kappa \sim U[0, \bar{\kappa}]$. In this case, the price of the loans at $t = 1$ given θ_1 is

$$q_1(\theta_1, 0) = q(\theta_2(\theta_1), 0) = \frac{R^2}{\bar{\kappa}(2 - \delta)} \theta_1,$$

so we get that

$$\theta_1 = \frac{\theta_0}{2 - \delta}.$$

Substituting θ_1 in $\hat{\Pi}_1(\theta_1, 0, \theta_0)$ and $q_1(\theta_1, 0)$, we get that

$$\Pi_1^E(\theta_0) = \frac{R^2}{2\bar{\kappa}} \frac{\theta_0^2}{(2-\delta)^2}.$$

and

$$q_0(\theta_0) = q_1(\theta_1(\theta_0), 0) = \frac{R^2}{\bar{\kappa}} \frac{\theta_0}{(2-\delta)^2}$$

In sum, the equilibrium loan selling strategy and loan prices at time $t = 1, 2$ are

$$\begin{aligned}
\theta_t &= \frac{\theta_{t-1}}{2-\delta} \\
q_t^E(\theta_t) &= \frac{R^2}{\bar{\kappa}(2-\delta)^{2-t}} \theta_t
\end{aligned}$$

Short-term Debt Financed bank With $D_1 > 0$, we formulate the Lagrangian and get the following first-order conditions for θ_1 and D_1:

$$\begin{aligned}
\frac{\partial \mathcal{L}_1}{\partial D_1} &= 1 - \delta - \eta = 0 \Rightarrow \eta = 1 - \delta. \\
\frac{\partial \mathcal{L}_1}{\partial \theta_1} &= -q_1(\theta_1, D_1) + \frac{\partial q_1(\theta_1, D_1)}{\partial \theta_1}(\theta_0 - \theta_1) + (\delta + \eta)\frac{\partial \Pi_2(\theta_1)}{\partial \theta_1} \\
&= -q_1(\theta_1, D_1) + \frac{\partial q_1(\theta_1, D_1)}{\partial \theta_1}(\theta_0 - \theta_1) + (\delta + \eta) q_2(\theta_2, D_2) \\
&= -q_1(\theta_1, D_1) + \frac{\partial q_1(\theta_1, D_1)}{\partial \theta_1}(\theta_0 - \theta_1) + (\delta + \eta) q_1(\theta_1, D_1) \\
&= \frac{\partial q_1(\theta_1, D_1)}{\partial \theta_1}(\theta_0 - \theta_1),
\end{aligned}$$

where the second line follows the envelope condition. From $\frac{\partial \mathcal{L}_1}{\partial \theta_1} = 0$, we know $\theta_1 = \theta_0$. Moreover, as $\eta > 0$, we find that the constraint $D_1 \leq \Pi_2(\theta_1)$ always binds. Therefore, given any θ_0, it is always the case that $\theta_1 = \theta_0$ and $D_1 = \Pi_1(\theta_1)$. The bank payoff is

$$\Pi_1^S(\theta_0) = \hat{\Pi}_1\left(\theta_0, \Pi_2^S(\theta_1), \theta_0\right) = \Pi_2^S(\theta_1),$$

so in the example with $b = 0$ and $\kappa \sim U[0, \bar{\kappa}]$ we get

$$\Pi_1^S(\theta_0) = \frac{R^2}{2\bar{\kappa}} \frac{\theta_0^2}{(2-\delta)}.$$

B E-bank equilibrium

In this section, we prove Proposition 1.

Lemma 4. *The bank with retention θ never sells a fraction of the loans.*

Proof. Suppose the bank with retention θ sells $\theta - \theta^+$, where $\theta^+ > 0$, and that after this it continues to trade smoothly. Multiple jumps are ruled out without loss of generality. In this case, the overall trading gains are $dG(\theta) + \Pi_E(\theta^+) - \Pi_E(\theta) = (\theta - \theta^+) q_E(\theta^+) + \Pi_E(\theta^+) - \Pi_E(\theta)$, where $dG(\theta)$ is the instant trading gain and $\Pi(\theta^+) - \Pi(\theta)$ are the gains (negative loss) in its continuation value. Block trading is suboptimal as long as

$$\theta = \arg\max_{\theta^+} \Big\{ \Pi_E(\theta^+) + (\theta - \theta^+) q_E(\theta^+) \Big\}. \tag{32}$$

It is easy to verify that the first order condition is always satisfied at $\theta^+ = \theta$, thus it suffice to show that the second order condition for global optimality is satisfied. $\square$

Verification of Optimality Trading Strategy

In this section, we complete the characterization of the equilibrium by verifying that the equilibrium trading strategy maximizes the bank's payoff given the price function $q_E(\theta)$. Throughout this section, we omit the subscript E to simplify notation. Hence, we denote the functions $\Pi_E(\theta), \pi_E(\theta), q_E(\theta)$ by $\Pi(\theta), \pi(\theta), q(\theta)$, respectively. Because the payoff in a mixed strategy equilibrium is given by the payoff of any pure strategy in its support, we can restrict attention to pure strategies in the verification of optimality. A trading strategy for the bank is given by the right continuous function with left limits. A trading strategy is admissible if it can be decomposed as

$$\theta_t = \int_0^t \dot{\theta}_s^c ds + \sum_{k \geq 0} (\theta_{t_k}^d - \theta_{t_k -}^d),$$

for some bounded function $\dot{\theta}_t^c$. We denote the set of admissible trading strategies by Θ. The bank's optimization problem is to choose $\theta \in \Theta$ to maximize its payoff

$$\Pi^*(\theta_0) = \sup_{\theta \in \Theta} \int_0^\infty e^{-(\rho+\phi)t} \big(\phi \pi(\theta_t) - \dot{\theta}_t^c q(\theta_t) \big) dt - \sum_{k \geq 0} e^{-(\rho+\phi)t_k} q(\theta_{t_k}) (\theta_{t_k}^d - \theta_{t_k -}^d). \tag{33}$$

Due to the discontinuity in the price function $q(\theta)$, the Hamilton-Jacobi-Bellman (HJB) equation is discontinuous at θ_*, so we need to resort to the theory of viscosity solutions for the analysis of the bank's problem. Our problem is a particular case of the general class of optimal control problems in stratified domains studied by Barles et al. (2018). Our proof relies on their characterization of the value function using viscosity solution methods. The analysis in Barles et al. (2018) does not consider the case in which the trajectory of the state variable can be discontinuous (impulse control). However, as we show below, we can approximate a trading $\theta_t \in \Theta$ by an absolutely continuous trading strategy with derivative $|\dot{\theta}_t| \leq N$ for some N large enough (the approximation is in the sense that it yields a similar payoff). Thus, we can consider a sequence of optimization problems

$$\Pi_N^*(\theta_0) = \sup_{|\dot{\theta}_t| \leq N} \int_0^\infty e^{-(\rho+\phi)t} \big(\phi \pi(\theta_t) - \dot{\theta}_t q(\theta_t) \big) dt, \tag{34}$$

and verify that, for any $\theta \in [0,1]$, $\Pi_N^*(\theta) \to \Pi(\theta)$, where

$$\Pi(\theta) = \begin{cases} \Phi\pi(\theta) & \text{if } \theta \in [\theta_*, 1] \\ q(0)\theta & \text{if } \theta \in [0, \theta_*) \end{cases}.$$

The following Lemma establishes that we can indeed consider the limit of bounded absolutely continuous strategies.

Lemma 5. *For any $\theta_0 \in [0,1]$, $\lim_{N\to\infty} \Pi_N^*(\theta_0) = \Pi^*(\theta_0)$.*

Proof. Let $\theta_t^{\epsilon *}$ be and ϵ-optimal policy (at this point in the proof, we have not established the existence of an optimal policy). For any $k \geq 0$, let $\Delta_k \equiv \inf\{\Delta > 0 : \theta^{\epsilon *}_{t_k-\Delta} + \operatorname{sgn}(\theta^{\epsilon *}_{t_k} - \theta^{\epsilon *}_{t_k-})N\Delta = \theta^{\epsilon *}_{t_k}\}$ (we can find Δ_k if N is large enough as $|\dot{\theta}_t^{\epsilon c *}| \leq M$ for some finite M). Consider the policy $\hat{\theta}_t^N = \theta_t^{\epsilon *}$ if $t \notin \cup_{k\geq 0}(t_k - \Delta_k, t_k)$, and $\hat{\theta}_t^N = \theta^{\epsilon *}_{t_k - \Delta_k} + \operatorname{sgn}(\theta^{\epsilon *}_{t_k} - \theta^{\epsilon *}_{t_k-})N(t - t_k + \Delta_k)$ if $t \in \cup_{k\geq 0}(t_k - \Delta_k, t_k)$. The difference between the payoff of $\theta_t^{\epsilon *}$ and $\hat{\theta}_t^N$ is

$$\begin{aligned}
\Pi^{\epsilon *}(\theta_0) - \hat{\Pi}_N &= \sum_{k\geq 0} \Bigg\{ \int_{t_k-\Delta_k}^{t_k} e^{-(\rho+\phi)t}\big(\phi\pi(\theta_t^{\epsilon *}) - \dot{\theta}_t^{\epsilon c *} q(\theta_t^{\epsilon *}) - \phi\pi(\hat{\theta}_t^N)\big)dt \\
&\quad + \int_{t_k-\Delta_k}^{t_k} e^{-(\rho+\phi)t}\operatorname{sgn}(\theta^{\epsilon *}_{t_k} - \theta^{\epsilon *}_{t_k-})Nq(\hat{\theta}_t^N)dt - e^{-(\rho+\phi)t_k} q(\theta^{\epsilon *}_{t_k})(\theta^{\epsilon d *}_{t_k} - \theta^{\epsilon d *}_{t_k-}) \Bigg\} - \epsilon \\
&= \sum_{k\geq 0} \Bigg\{ \int_{t_k-\Delta_k}^{t_k} e^{-(\rho+\phi)t}\big(\phi\pi(\theta_t^{*}) - \dot{\theta}_t^{\epsilon c *} q(\theta_t^{*}) - \phi\pi(\hat{\theta}_t^N)\big)dt \\
&\quad + \frac{\theta^{\epsilon *}_{t_k} - \theta^{\epsilon *}_{t_k-\Delta_k}}{\Delta_k} \int_{t_k-\Delta_k}^{t_k} e^{-(\rho+\phi)t} q(\hat{\theta}_t^N)dt - e^{-(\rho+\phi)t_k} q(\theta^{\epsilon *}_{t_k})(\theta^{\epsilon d *}_{t_k} - \theta^{\epsilon d *}_{t_k-}) \Bigg\} - \epsilon.
\end{aligned}$$

For all $k \geq 0$, we have that $\Delta_k \downarrow 0$ as $N \to \infty$. It follows that

$$\lim_{\Delta_k \downarrow 0} \frac{1}{\Delta_k} \int_{t_k-\Delta_k}^{t_k} e^{-(\rho+\phi)t} q(\hat{\theta}_t^N)dt = \begin{cases} e^{-(\rho+\phi)t_k} q(\theta^{\epsilon *}_{t_k}-) & \text{if } \theta^{\epsilon *}_{t_k} > \theta^{\epsilon *}_{t_k-} \\ e^{-(\rho+\phi)t_k} q(\theta^{\epsilon *}_{t_k}+) & \text{if } \theta^{\epsilon *}_{t_k} < \theta^{\epsilon *}_{t_k-}. \end{cases}$$

The price function is right continuous so $q(\theta^{\epsilon *}_{t_k}+) = q(\theta^{\epsilon *}_{t_k})$. We can conclude that

$$\lim_{N\to\infty} (\Pi^{\epsilon *}(\theta_0) - \hat{\Pi}_N(\theta_0)) = \sum_{k\geq 0} e^{-(\rho+\phi)t_k}\big(q(0) - q(\theta_*)\big)(\theta^*_{t_k} - \theta^*_{t_k-})^+ \mathbf{1}_{\{\theta^*_{t_k} = \theta_*\}} - \epsilon \leq 0.$$

Because this holds for any $\epsilon > 0$, we can conclude that $\lim_{N\to\infty}(\Pi^*(\theta_0) - \hat{\Pi}_N(\theta_0)) \leq 0$, and given that $\Pi^*(\theta_0) \geq \hat{\Pi}_N(\theta_0)$, we get $\lim_{N\to\infty} \hat{\Pi}_N(\theta_0) = \Pi^*(\theta_0)$. For N large enough, the policy $\hat{\theta}_t^N$ satisfies $|\dot{\hat{\theta}}_t^N| \leq N$ (this can be guarantee because for any ϵ there is M such that $|\dot{\theta}_t^{\epsilon c *}| \leq M$), so its payoff, $\hat{\Pi}_N(\theta_0)$ provides a lower bound to $\Pi_N^*(\theta_0)$, which means that $\lim_{N\to\infty} \Pi_N^*(\theta_0) = \Pi^*(\theta_0)$. ◻

This shows that the value function converges (pointwise) to the one in the equilibrium under consideration. Hence, we can verify the optimality of the bank's strategy by analyzing the control problem (34). For future reference, recall that the price function in the control problem (34) is given by

$$q(\theta)=\begin{cases}\Phi\pi'(\theta) & \text{if } \theta\geq\theta_* \\ p_L R\,(1-\alpha) & \text{if } \theta<\theta_*\end{cases}, \tag{35}$$

where the threshold θ_* is given by $\Phi\pi(\theta_*)=q(0)\theta_*$. Notice that we are not computing the equilibrium in a model in which the bank is restricted to use absolutely continuous trading strategies with bounded derivative $\dot{\theta}_t$, but rather considering the equilibrium price function in the general case, and then considering a sequence of auxiliary optimization problems to construct the value function. Because the expected payoff of the candidate equilibrium strategy is equal to the value function, it is necessarily optimal.

The Hamilton-Jacobi-Bellman equation (HJB) for the optimization problem (34) is

$$(\rho+\phi)\Pi_N(\theta)-H(\theta,\Pi_N'(\theta))=0, \tag{36}$$

where H

$$H(\theta,\Pi_N')\equiv\phi\pi(\theta)+\max_{|\dot{\theta}|\leq N}\left\{\dot{\theta}\big(\Pi_N'-q(\theta)\big)\right\}. \tag{37}$$

We guess and verify that, for N large enough, the solution (in the viscosity sense) of the previous equation is

$$\Pi_N(\theta)=\begin{cases}\Phi\pi(\theta) & \text{if } \theta\in[\theta_*,1] \\ e^{-\frac{\rho+\phi}{N}(\theta_*-\theta)}\Phi\pi(\theta_*)+\frac{(\rho+\phi)}{N}\int_\theta^{\theta_*}e^{-\frac{\rho+\phi}{N}(y-\theta)}\left(\Phi\pi(y)-\frac{N}{\rho+\phi}q(0)\right)dy & \text{if } \theta\in[\tilde{\theta}_N,\theta_*) \\ \frac{N}{\rho+\phi}\left(1-e^{-\frac{(\rho+\phi)}{N}\theta}\right)q(0)+\frac{(\rho+\phi)}{N}\int_0^\theta e^{-\frac{(\rho+\phi)}{N}(\theta-y)}\Phi\pi(y)dy & \text{if } \theta\in[0,\tilde{\theta}_N),\end{cases} \tag{38}$$

where $\tilde{\theta}_N$ is the unique solution on $[0,1]$ to the equation

$$\frac{N}{\rho+\phi}\left(1-e^{-\frac{(\rho+\phi)}{N}\tilde{\theta}_N}\right)q(0)+\frac{(\rho+\phi)}{N}\int_0^{\tilde{\theta}_N}e^{-\frac{(\rho+\phi)}{N}(\tilde{\theta}_N-y)}\Phi\pi(y)dy= \\ e^{-\frac{\rho+\phi}{N}(\theta_*-\tilde{\theta}_N)}\Phi\pi(\theta_*)+\frac{(\rho+\phi)}{N}\int_{\tilde{\theta}_N}^{\theta_*}e^{-\frac{\rho+\phi}{N}(y-\tilde{\theta}_N)}\left(\Phi\pi(y)-\frac{N}{\rho+\phi}q(0)\right)dy \tag{39}$$

B.1 Auxiliary Lemmas

Before proceeding with the verification theorem, we provide several Lemmas providing properties of our candidate value function $\Pi_N(\theta)$ that will be later used in the verification

Lemma 6. *If $\Phi\pi_E(1)>p_L R_o>0$, then there exists a unique $\theta_*\in(0,1)$ solving the equation*

$$\theta_* q_E(0)=\Phi\pi_E(\theta_*) \tag{40}$$

If $p_L=0$, then $\theta_*=0$ is the unique solution to (40) on $[0,1]$.

Proof. As $\Phi\pi_E(0) = 0$, equation (40) is trivially satisfied at $\theta_* = 0$, we want to show that if $\Phi\pi_E(1) > p_L R_o = q_E(0)$, then there is a non trivial solution $\theta_* > 0$ that also satisfies equation (40). First, if $\Phi\pi_E(1) > p_L R_o$, then the right hand side of equation (40) is strictly larger than its left hand side evaluated at $\theta_* = 1$. Second, as $\Phi\pi_E'(0) < q_E(0)$ it follows that for ε small enough $\varepsilon q_E(0) > \Phi\pi_E(\varepsilon)$. Thus, it follows from continuity that a nontrivial solution exists on $(0,1)$. Uniqueness follows because

$$\begin{aligned}
q_E(0) - \Phi\pi_E'(\theta_*) &= \frac{\Phi\pi_E(\theta_*)}{\theta_*} - \Phi p_E(\theta_*) R_o \\
&= \Phi\left[p_E(\theta_*) R_o - \frac{1}{\theta_*}\int_0^{\kappa_E(\theta_*)} \kappa dF(\kappa)\right] - \Phi p_E(\theta_*) R_o < 0,
\end{aligned}$$

so the function $\theta q_E(0) - \Phi\pi_E(\theta)$ single crosses 0 from above, which implies $\theta q_E(0) > \Phi\pi_E(\theta)$ on $\theta \in (0,\theta_*)$ and $\theta q_E(0) < \Phi\pi_E(\theta)$ on $\theta \in (\theta_*, 1]$. Finally, if $p_L = 0$, then $\Phi\pi_E'(0) = q_E(0) = 0$. It follows then from the convexity of $\pi_E(\theta)$ that $\theta_* = 0$ is a global maximum of $\theta q_E(0) - \Phi\pi_E(\theta)$, which means that $\theta q_E(0) < \Phi\pi_E(\theta)$ for all $\theta > 0$. ☐

Lemma 7. *There is a unique solution $\tilde{\theta}_N \in (0,\theta_*)$ to equation (39).*

Proof. First, we show existence. Given the definition of θ_* and the convexity of $\pi(\theta)$ we have that $\Phi\pi(\theta) < \theta q(0)$ for all $\theta < \theta_*$. Hence,

$$\frac{N}{\rho+\phi}\left(1 - e^{-\frac{(\rho+\phi)}{N}\theta_*}\right) q(0) + \frac{(\rho+\phi)}{N}\int_0^{\theta_*} e^{-\frac{(\rho+\phi)}{N}(\theta-y)}\Phi\pi(y)dy < \Phi\pi(\theta_*).$$

We also have that

$$\begin{aligned}
e^{-\frac{\rho+\phi}{N}\theta_*}\Phi\pi(\theta_*) + \frac{(\rho+\phi)}{N}\int_0^{\theta_*} e^{-\frac{\rho+\phi}{N}y}\left(\Phi\pi(y) - \frac{N}{\rho+\phi}q(0)\right)dy &\leq \Phi\pi(\theta_*) - \frac{N}{(\rho+\phi)}\left(1 - e^{-\frac{\rho+\phi}{N}\theta_*}\right)q(0) \\
&\leq \Phi\pi(\theta_*) - \theta_* q(0) = 0.
\end{aligned}$$

The existence of a solution follows from the intermediate value theorem. To show uniqueness, we consider the derivative of the difference between the left and the right-hand sides of equation (39) evaluated at θ, which we denote by $G'(\theta)$.

$$\begin{aligned}
G'(\theta) = e^{-\frac{(\rho+\phi)}{N}\theta}q(0) + \frac{(\rho+\phi)}{N}\Phi\pi(\theta) - \frac{(\rho+\phi)^2}{N^2}\int_0^{\theta} e^{-\frac{(\rho+\phi)}{N}(\theta-y)}\Phi\pi(y)dy \\
- \frac{(\rho+\phi)}{N}e^{-\frac{\rho+\phi}{N}(\theta_*-\theta)}\Phi\pi(\theta_*) + \frac{(\rho+\phi)}{N}\left(\Phi\pi(\theta) - \frac{N}{\rho+\phi}q(0)\right) \\
- \frac{(\rho+\phi)^2}{N^2}\int_\theta^{\theta_*} e^{-\frac{\rho+\phi}{N}(y-\theta)}\left(\Phi\pi(y) - \frac{N}{\rho+\phi}q(0)\right)dy
\end{aligned}$$

From here we get that

$$
\begin{aligned}
G'(\theta_*) &= -\left(1-e^{-\frac{(\rho+\phi)}{N}\theta_*}\right)q(0)-\frac{(\rho+\phi)^2}{N^2}\int_0^{\theta_*}e^{-\frac{(\rho+\phi)}{N}(\theta_*-y)}\Phi\pi(y)dy+\frac{(\rho+\phi)}{N}\Phi\pi(\theta_*)\\
&\leq \frac{(\rho+\phi)}{N}\left(\Phi\pi(\theta_*)-\theta_* q(0)\right)=0\\
G'(0) &= -\frac{(\rho+\phi)}{N}e^{-\frac{\rho+\phi}{N}\theta_*}\Phi\pi(\theta_*)-\frac{(\rho+\phi)^2}{N^2}\int_0^{\theta_*}e^{-\frac{\rho+\phi}{N}y}\left(\Phi\pi(y)-\frac{N}{\rho+\phi}q(0)\right)dy\\
&\geq -\frac{(\rho+\phi)}{N}\Phi\pi(\theta_*)+\left(1-e^{-\frac{\rho+\phi}{N}\theta_*}\right)q(0)\\
&\geq \frac{(\rho+\phi)}{N}\left(\theta_* q(0)-\Phi\pi(\theta_*)\right)=0
\end{aligned}
$$

Moreover, we get that, for any $\theta \in (0,\theta_*)$, $G(\theta)=0$ implies

$$
\begin{aligned}
G'(\theta) &= \frac{2(\rho+\phi)}{N}\left[\Phi\pi(\theta)-\left(\frac{N}{\rho+\phi}\left(1-e^{-\frac{(\rho+\phi)}{N}\theta}\right)q(0)+\frac{(\rho+\phi)}{N}\int_0^{\theta}e^{-\frac{(\rho+\phi)}{N}(\theta-y)}\Phi\pi(y)dy\right)\right]\\
&\leq \frac{2(\rho+\phi)}{N}\left[\Phi\pi(\theta)-\theta q(0)\right]<0.
\end{aligned}
$$

It follows that $G(\theta)$ single crosses 0, so there is a unique solution to the equation $G(\theta)=0$. ☐

Lemma 8. *There is $\tilde{N}$ such that, for all $N>\tilde{N}$, $\Pi'_N(\theta)<q(0)$ on $(0,\tilde{\theta}_N)$ and $\Pi'_N(\theta)>q(0)$ on $(\tilde{\theta}_N,\theta_*)$.*

Proof. First, we very that $\Pi'_N(\theta)<q(0)$ on $(0,\tilde{\theta}_N)$. The derivative of $\Pi_N(\theta)-\theta q(0)$ on $(0,\tilde{\theta}_N)$ is given by

$$
\begin{aligned}
\Pi'_N(\theta)-q(0) &= \frac{\rho+\phi}{N}\Phi\pi(\theta)-\left(1-e^{-\frac{(\rho+\phi)}{N}\theta}\right)q(0)-\frac{(\rho+\phi)^2}{N^2}\int_0^{\theta}e^{-\frac{(\rho+\phi)}{N}(\theta-y)}\Phi\pi(y)dy\\
&\leq \frac{\rho+\phi}{N}\Phi\pi(\theta)-\left(1-e^{-\frac{(\rho+\phi)}{N}\theta}\right)q(0)\leq \frac{\rho+\phi}{N}\left(\Phi\pi(\theta)-\theta q(0)\right)<0.
\end{aligned}
$$

The derivative of $\Pi_N(\theta)-\theta q(0)$ on $(\tilde{\theta}_N,\theta_*)$ is given by

$$
\Pi'_N(\theta)-q(0)=\frac{\rho+\phi}{N}\left[e^{-\frac{\rho+\phi}{N}(\theta_*-\theta)}\Phi\pi(\theta_*)-\Phi\pi(\theta)+\frac{(\rho+\phi)}{N}\int_\theta^{\theta_*}e^{-\frac{\rho+\phi}{N}(y-\theta)}\left(\Phi\pi(y)-\frac{N}{\rho+\phi}q(0)\right)dy\right]
$$

Differentiating the HJB equation, we get that

$$
\begin{aligned}
\Pi''_N(\theta) &= \frac{(\rho+\phi)}{N}\left(\Pi'_N(\theta)-\Phi\pi'(\theta)\right)\\
\Pi'''_N(\theta) &= \frac{(\rho+\phi)}{N}\left(\Pi''_N(\theta)-\Phi\pi''(\theta)\right).
\end{aligned}
$$

From here we get that $\Pi''_N(\theta)=0 \implies \Pi'''_N(\theta)<0$, so we the function $\Pi'_N(\theta)$ is quasi-concave on $(\tilde{\theta}_N,\theta_*)$.

Moreover, $\Pi'_N(\theta_*-) = q(0)$, and

$$\Pi'_N(\tilde{\theta}_N+) = q(0) + \frac{(\rho+\phi)}{N}\Big(\Pi_N(\tilde{\theta}_N+) - \Phi\pi(\tilde{\theta}_N)\Big) > q(0),$$

so we can conclude that $\Pi'_N(\theta) > q(0)$ on $(\tilde{\theta}_N, \theta_*)$ as long as $\Pi_N(\tilde{\theta}_N+) > \Phi\pi(\tilde{\theta}_N)$, which follows from

$$\begin{aligned}
\Pi_N(\tilde{\theta}_N+) - \Phi\pi(\tilde{\theta}_N) &= \Pi_N(\tilde{\theta}_N-) - \Phi\pi(\tilde{\theta}_N) \\
&= \frac{N}{\rho+\phi}\Big(1 - e^{-\frac{(\rho+\phi)}{N}\tilde{\theta}_N}\Big) q(0) - \Phi\pi(\tilde{\theta}_N) + \frac{(\rho+\phi)}{N}\int_0^{\tilde{\theta}_N} e^{-\frac{(\rho+\phi)}{N}(\tilde{\theta}_N - y)}\Phi\pi(y)dy \\
&\geq \frac{N}{\rho+\phi}\Big(1 - e^{-\frac{(\rho+\phi)}{N}\tilde{\theta}_N}\Big) q(0) - \Phi\pi(\tilde{\theta}_N) \\
&= \tilde{\theta}_N q_0 - \frac{(\rho+\phi)\tilde{\theta}_N^2}{N} - \Phi\pi(\tilde{\theta}_N) + O\big(1/N^2\big).
\end{aligned}$$

$\tilde{\theta}_N q_0 > \Phi\pi(\tilde{\theta}_N)$ because θq_0 single crosses $\Phi\pi(\theta)$ at $\theta_* \geq \tilde{\theta}_N$. Hence, there is $\tilde{N}$ such that, for all $N \geq \tilde{N}$, we have $\Pi_N(\tilde{\theta}_N+) > \Phi\pi(\tilde{\theta}_N)$. □

Lemma 9. *Let*

$$\Pi(\theta) = \begin{cases} \Phi\pi(\theta) & \text{if } \theta \in [\theta_*, 1] \\ q(0)\theta & \text{if } \theta \in [0, \theta_*) \end{cases}.$$

Then, for any $\theta \in [0,1]$

$$\lim_{N\to 0} \Pi_N(\theta) = \Pi(\theta).$$

Proof. For all $\theta \geq \theta_*$, $\Pi_N(\theta) = \Pi(\theta)$, and, for any $\theta < \theta_*$, $\lim_{N\to\infty} \Pi_N(\theta) = \theta q(0) = \Pi(\theta)$ by L'Hopital's rule. □

B.2 Verification of Optimality

We start providing the necessary definitions from the theory of viscosity solutions, together with the relevant results from the theory of optimal control in stratified domains in Barles et al. (2018). We make some changes in notation to make it consistent with our setting, and to translate their minimization problem into a maximization one. While Barles et al. (2018) considers the state space to be the complete real line, the state space in our case is $[0,1]$. However, we can extend the state space by letting the payoff on the complement of $[0,1]$ be sufficiently low. This can be achieved by adding a penalization term and setting the flow payoff equal to $\phi\pi(1) - \dot{\theta}q(1) - k|\theta - 1|$ for $\theta > 1$, and $\phi\pi(0) - \dot{\theta}q(0) - k|\theta|$ for $\theta < 0$. By choosing k large enough, we can ensure that the optimal solution never exits the interval $[0,1]$. Due to the discontinuity in the Hamiltonian at θ_*, a viscosity solution might fail to be unique. In order to fully characterize the value function we need to specify its behavior at θ_*. This is done in Barles et al. (2018) by considering the concept of *Flux-limited* sub- and supersolutions. Letting $\Omega_0 = (-\infty, \theta_*)$ and $\Omega_1 = (\theta_*, \infty)$, we consider the equation

$$\begin{cases} (\rho+\phi)\Pi - H_0(\theta, \Pi) = 0 & \text{in } \Omega_0 \\ (\rho+\phi)\Pi - H_1(\theta, \Pi) = 0 & \text{in } \Omega_1 \\ (\rho+\phi)\Pi - \phi\pi(\theta_*) = 0 & \text{in } \{\theta_*\}, \end{cases} \tag{41}$$

where

$$H_0(\theta, \Pi') = \phi\pi(\theta) + k\min\{0, \theta\} + \max_{|\dot\theta|\le N}\left\{\dot\theta\big(\Pi' - q(0)\big)\right\}$$
$$H_1(\theta, \Pi') = \phi\pi(\theta) - k\max\{0, \theta-1\} + \max_{|\dot\theta|\le N}\left\{\dot\theta\big(\Pi' - q(\theta)\big)\right\}$$

In $\Omega_0 \cup \Omega_1$, the definitions are just classical viscosity sub- and supersolution, which we provide next for completeness.

Definition 3 (Bardi and Capuzzo-Dolcetta (2008), Definition 1.1)**.** *A function $u \in C(\mathbb{R})$ is a viscosity subsolution of (36) if, for any $\varphi \in C^1(\mathbb{R})$,*

$$(\rho+\phi)u(\theta_0) - H(\theta_0, \varphi'(\theta_0)) \le 0, \tag{42}$$

at any local maximum point $\theta_0 \in \mathbb{R}$ of $u - \varphi$. Similarly, $u \in C(\mathbb{R})$ is a viscosity supersolution of (36) if, for any $\varphi \in C^1(\mathbb{R})$,

$$(\rho+\phi)u(\theta_1) - H(\theta_1, \varphi'(\theta_1)) \ge 0, \tag{43}$$

at any local minimum point $\theta_1 \in \mathbb{R}$ of $u - \varphi$. Finally, u is a viscosity solution of (36) if it is simultaneously a viscosity sub- and supersolution.

Before providing the definition of sub- and supersolution on $\{\theta_*\}$, we introduce the following space $\Im$ of real valued test functions: $\varphi \in \Im$ if $\varphi \in C(\mathbb{R})$ and there exist $\varphi_0 \in C^1(\overline{\Omega}_0)$ and $\varphi_1 \in C^1(\overline{\Omega}_1)$ such that $\varphi = \varphi_0$ in $\overline{\Omega}_0$, and $\varphi = \varphi_1$ in $\overline{\Omega}_1$. Next, we introduce two Hamiltonians that are needed to define a flux-limited sub- and supersolution at $\{\theta_*\}$.

$$H_1^+(\theta_*, \Pi') \equiv \phi\pi(\theta) + \sup_{0<\dot\theta\le N}\left\{\dot\theta\big(\Pi' - q(\theta_*)\big)\right\}$$
$$H_0^-(\theta_*, \Pi') \equiv \phi\pi(\theta) + \sup_{0>\dot\theta\ge -N}\left\{\dot\theta\big(\Pi' - q(0)\big)\right\}.$$

Definition 4 (Barles et al. (2018), Definition 2.1)**.** *An upper semi-continuous, bounded function $u : \mathbb{R} \to \mathbb{R}$ is a flux-limited subsolution on $\{\theta_*\}$ if for any test function $\varphi \in \Im$ such that $u - \varphi$ has a local maximum at θ_*, we have*

$$(\rho+\phi)u(\theta_*) - \max\{\phi\pi(\theta_*), H_0^-(\theta_*, \varphi_0'(\theta_*)), H_1^+(\theta_*, \varphi_1'(\theta_*))\} \le 0. \tag{44}$$

A lower semi-continuous, bounded function $v : \mathbb{R} \to \mathbb{R}$ is a flux-limited supersolution on $\{\theta_*\}$ if for any test *function $\varphi \in \Im$ such that $u - \varphi$ has a local minimum at θ_*, we have*

$$(\rho+\phi)v(\theta_*) - \max\{\phi\pi(\theta_*), H_0^-(\theta_*, \varphi_0'(\theta_*)), H_1^+(\theta_*, \varphi_1'(\theta_*))\} \geq 0. \tag{45}$$

The Hamiltonians H_0^- and H_1^+ are needed to express the optimality conditions at the discontinuity θ_*. H_1^+ consider controls that starting at θ_* take θ_t towards the interior of $[\theta_*, 1]$, and H_0^- considers controls that starting at θ_* take θ_t towards the interior of $[0, \theta_*]$. The use of the Hamiltonians H_0^- and H_1^+ at $\{\theta_*\}$, instead of H_0 and H_1, distinguishes *flux-limited* viscosity solutions from the traditional (discontinuous) viscosity solutions.

We consider the following control problem, equivalent to the one defined in (34),

$$\begin{aligned}
\tilde{\Pi}_N^*(\theta_0) &= \sup_{|\theta_t|\leq N} \int_0^\infty e^{-(\rho+\phi)t}\Big(\phi\tilde{\pi}(\theta_t) - \theta_t\tilde{q}(\theta_t)\mathbf{1}_{\{\theta_t\neq\theta_*\}} - k\big(\max\{0, \theta_t - 1\} - \min\{0, \theta_t\}\big)\Big)dt \\
\tilde{\pi}(\theta) &= \pi(0)\mathbf{1}_{\{\theta<0\}} + \pi(\theta)\mathbf{1}_{\{\theta\in[0,1]\}} + \pi(1)\mathbf{1}_{\{\theta>1\}} \\
\tilde{q}(\theta) &= q(0)\mathbf{1}_{\{\theta<0\}} + q(\theta)\mathbf{1}_{\{\theta\in[0,1]\}} + q(1)\mathbf{1}_{\{\theta>1\}}.
\end{aligned}$$

The following Theorem characterizes the value function $\tilde{\Pi}_N^*$ in terms of flux-limited viscosity solutions.

Theorem 1 (Barles et al. (2018), Theorem 2.9)**.** *The value function $\tilde{\Pi}_N^*$ is the unique flux-limited viscosity solution to equation (41).*

We can now proceed to apply Theorem 1 to verify that Π_N defined in (38) is the value function of the control problem (34).

Verification Lemmas 7 and 8 imply that Π_N is a classical solution on $\Omega \setminus \{\tilde{\theta}_N, \theta_*\}$ so we only need to verify the conditions for a viscosity solution on $\{\tilde{\theta}_N, \theta_*\}$. Π_N defined in (38) is a classical solution on $(\theta_*, 1)$. At $\theta = \theta_*$, Π_N has a convex kink, so we only need to verify the supersolution property. That is, that for any $\varphi'(\theta_*)$ in the subdifferential of $\Pi_N(\theta)$ at θ_*, which is $[\Pi_N'(\theta_*-), \Pi_N'(\theta_*+)]$, inequality (45) is satisfied. $H_1^+\big(\theta_*, \varphi'(\theta_*)\big)$ is nondecreasing in $\varphi'(\theta_*)$ and $H_0^-\big(\theta_*, \varphi'(\theta_*)\big)$ is nonincreasing in $\varphi'(\theta_*)$; thus, the supersolution property follows from

$$\begin{aligned}
(\rho+\phi)\Pi_N(\theta_*) - H_1^+\big(\theta_*, \Pi_N'(\theta_*+)\big) &= (\rho+\phi)\Pi_N(\theta_*) - \phi\pi(\theta_*) = 0 \\
(\rho+\phi)\Pi_N(\theta_*) - H_0^-\big(\theta_*, \Pi_N'(\theta_*-)\big) &= (\rho+\phi)\Pi_N(\theta_*) - \phi\pi(\theta_*) = 0.
\end{aligned}$$

As $\Pi_N'(\tilde{\theta}_N-) < q(0) < \Pi_N'(\tilde{\theta}_N+)$, $\Pi_N(\theta)$ has a convex kink at $\tilde{\theta}_N$, we only need to verify the property for a supersolution. Thus, we need to verify that for any $\varphi'(\tilde{\theta}_N) \in [\Pi_N'(\tilde{\theta}_N-), \Pi_N'(\tilde{\theta}_N+)]$, inequality (43) is satisfied. This amounts to verifying that

$$(\rho+\phi)\Pi_N(\tilde{\theta}_N) - \phi\pi(\tilde{\theta}_N) \geq N\max\Big\{\big|\Pi_N'(\tilde{\theta}_N-) - q(0)\big|, \big|\Pi_N'(\tilde{\theta}_N+) - q(0)\big|\Big\}$$

By definition of $\tilde{\theta}_N$, we have

$$(\rho+\phi)\Pi_N(\tilde{\theta}_N) - \phi\pi(\tilde{\theta}_N) = N\big(\Pi'_N(\tilde{\theta}_N+) - q(0)\big) = N\big(q(0) - \Pi'_N(\tilde{\theta}_N-)\big),$$

so it follows that

$$(\rho+\phi)\Pi_N(\tilde{\theta}_N) - \phi\pi(\tilde{\theta}_N) = N\big|\Pi'_N(\tilde{\theta}_N-) - q(0)\big| = N\big|\Pi'_N(\tilde{\theta}_N+) - q(0)\big|.$$

Finally, at $\theta = 1$, by choosing k large enough, we have that the solution of the HJB equation on $\{\theta > 1\}$ entails $\dot{\theta}(\theta) = -N$. Moreover, $\Pi'(1-) = q(1)$ implies that the value function is differentiable at $\theta = 1$ (in the extended problem) and that $\dot{\theta}(1) \leq 0$ is optimal. Thus, the state constraint is satisfied at $\theta = 1$. A similar argument applies at $\theta = 0$. Thus, we can conclude that $\Pi_N(\theta)$ is a flux-limited viscosity solution, so, by Theorem 1, it is the value function of the optimal control problem.

The uniqueness proof follows from the fact that the HJB has a unique solution in the smooth trading region, under the given boundary conditions. In the region $[0, \theta_*)$, smooth trading and no trading are both ruled out. This implies that trading must be atomistic. Under atomistic trading, the price must be at least $p_L R_o$, but the bank will sell everything, even below this price. The solution θ_* is easily verified as being unique.

C S-bank equilibrium

Proof of Lemma 1

Proof. Using integration by parts and a Transversality condition $\lim_{t\to\infty} e^{-(\rho+\phi)t} D_t = 0$, we have

$$\int_0^\infty e^{-(\rho+\phi)s} dD_s = \int_0^\infty e^{-(\rho+\phi)s}(\rho+\phi)D_s ds - D_0.$$

(20) can be rewritten as

$$\max_{(D_t,\theta_t)_{t\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-(\rho+\phi)s}\left[\left(\phi\hat{\pi}_S(\theta_s, D_s) - y\left(\theta_s, D_s\right) D_s + (\rho+\phi)D_s\right) ds + dG\left(\theta_s\right)\right]\right] - D_0$$

The optimization in the lemma follows directly from the definition of $\phi\pi(\theta)$ □

Proof of Lemma 2 and 3

Proof. Let $\mathcal{L} = \mathcal{V}(D,\theta) + \Phi z(\theta)\left[\Pi_S(\theta) - D\right]$, where $\Phi z(\theta)$ is the Lagrange multiplier of the debt issuance constraint. From the first order condition, $\mathcal{L}_D = 0$, we get that the optimal solution $D^*(\theta)$ solves

$$\phi\left[f(\kappa_S)\Delta^2\right] D^*(\theta) = \rho - \phi z(\theta) \Rightarrow z(\theta) = \left(\frac{\rho}{\phi} - \left[f(\kappa_S)\Delta^2\right]\Pi_S(\theta)\right)^+. \tag{46}$$

To finish the derivation of optimal debt choice $D^*(\theta)$, we need one more assumption to guarantee that the second-order condition of the constrained maximization problem is satisfied. Assumption 2 imposes lower and upper bounds on $f'(\kappa)$, and will be satisfied in the uniform-distribution case. The upper bound guarantees that, for any given θ, the second-order partial derivative of (23) will satisfy $\mathcal{L}_{DD} < 0$. The lower bound guarantees that, in the region where the equity holders' limited liability binds, the value function $\Pi_S(\theta)$ will be convex in θ.

Lemma 2 follows from the fact that $z'(\theta) < 0$, which is necessarily the case given Assumption 2 and equation (46). Moreover, the constraint clearly binds at $\theta = 0$ with $D^*(0) = \Pi_S(0) = 0$.

There are two cases to be considered: when the borrowing constraint is binding and when it is not. When the borrowing constraint is slack, then $z(\theta) = 0$, so the conclusion follows directly. Thus, we only need to verify the case in which the borrowing constraint, $D \leq \Pi_S(\theta)$, is binding.

Note that from the solution to $D^*(\theta)$, it is immediately clear that $z(\theta) \leq \frac{\rho}{\phi}$. In the constrained region, $\Pi_S(\theta) = \mathcal{V}(\Pi_S(\theta), \theta)$. Using the Envelope Theorem to solve for $\Pi'_S(\theta)$, we get

$$\begin{aligned}
\Pi'_S(\theta) &= \frac{d\mathcal{V}(\Pi_S(\theta), \theta)}{d\theta} \\
&= \left.\frac{\partial \mathcal{L}}{\partial \theta}\right|_{D=\Pi_S(\theta)} \\
&= \frac{\Phi}{1 - \Phi z(\theta)} \left[p(\theta) R_o + f(\kappa_S) \Delta^2 R_o \Pi_S(\theta)\right].
\end{aligned}$$

Since the *F.O.C.* implies

$$f(\kappa_S) \Delta^2 \Pi_S(\theta) = \frac{\rho - \phi z(\theta)}{\phi},$$

we can show

$$\Pi'_S(\theta) = \frac{\Phi}{1 - \Phi z(\theta)} \left[p(\theta) + \frac{\rho - \phi z(\theta)}{\phi}\right] R_o.$$

Substituting this expression in equation (27), we get

$$\begin{aligned}
\dot{\theta} &= \phi \frac{(1 - \Phi)(1 - p(\theta)) R_o + \Phi z(\theta) (\Pi'_S(\theta) - R_o)}{\Phi \pi''_S(\theta)} \\
&= \phi \frac{(1 - \Phi)(1 - p(\theta)) + \Phi z(\theta) \left(\phi \frac{p(\theta) - 1}{(\rho + \phi) - \phi z(\theta)}\right)}{\Phi \pi''_S(\theta)} R_o \\
&= \phi \frac{(1 - \Phi) - \frac{\Phi z(\theta) \phi}{(\rho + \phi) - \phi z(\theta)}}{\Phi \pi''_S(\theta)} (1 - p(\theta)) R_o.
\end{aligned}$$

Clearly, $\dot{\theta} > 0$ if and only if

$$(1 - \Phi) - \frac{\Phi z(\theta) \phi}{(\rho + \phi) - \phi z(\theta)} > 0 \Longrightarrow \frac{\rho}{\phi} > \frac{\phi}{(\rho + \phi) - \phi z(\theta)} z(\theta),$$

which always holds because of the upper bound on the Lagrange multiplier, $z(\theta) \leq \frac{\rho}{\phi}$.

□

Proof of Proposition 2

Proof. The convexity of $\Pi_S(\theta)$ in the unconstrained region $D^*(\theta) < \Pi_S(\theta)$ follows under Assumption 2. In the constrained region where $D^*(\theta) = \Pi_S(\theta)$, we differentiate $\Pi'_S(\theta)$ and get

$$\Pi''_S(\theta) = \frac{\Phi}{1-\Phi z(\theta)}\left[d'_S(\theta) + z'(\theta)\left(\Pi'_S(\theta) - R_o\right)\right].$$

Substituting the first order condition for D in $\Pi'_S(\theta)$, we get that

$$\Pi'_S(\theta) - R_o = -\frac{\Phi}{1-\Phi z(\theta)}\left(1-p(\theta)\right) < 0.$$

Hence, it suffices to show that $z'(\theta) \le 0$. Note that (46) implies

$$z'(\theta) = -\left[f'\Delta^2\Pi_S(\theta)\frac{\partial \kappa_S}{\partial \theta} + f\Delta^2\Pi'_S(\theta)\right].$$

Note that $\Pi'_S(\theta) = \Phi\pi'_S(\theta)$, so that

$$\Pi'_S(\theta) = \frac{\Phi}{1-\Phi z(\theta)}\left[p(\theta)R_o + f\Delta^2 R_o\Pi_S(\theta)\right].$$

Therefore, for $z'(\theta) < 0$, it suffices to have $\Delta f'\Pi_S(\theta) + \frac{\Phi}{1-\Phi z(\theta)}\left[p(\theta) + f\Delta^2\Pi_S(\theta)\right] > 0$, which always holds under the lower bound imposes by Assumption 2.

Next, at the boundary θ_D where the equity holder's limited liability constraint binds, $\Pi'_S(\theta)$ is continuous, which follows from (26), and the continuity of $d_S(\theta)$, $D^*(\theta)$, as well as the fact that $\lim_{\theta\uparrow\theta_D} z(\theta) = z(\theta_D) = 0$. Therefore, we conclude that $\Pi_S(\theta)$, the value function in the smooth-trading case is globally convex.

Finally, note that at $\theta = 0$, $\Pi'_S(\theta) < p_L R_o$. At $\theta = 1$, $\Pi_S(1) > \Pi_E(1)$, where $\Pi_E(1) > p_L R_o$ follows from Assumption 1. The intersection between $\Pi_S(\theta)$ and $p_L R_o\theta$ then admits the unique solution $\theta_\dagger$. Moreover, $\Pi_S(\theta) < p_L R_o\theta$ if and only if $\theta < \theta_\dagger$.

□

D Analysis of the Linear-Quadratic Example

We derive the results under $\theta_0 \in [0,1]$, and the results in subsection 5.2 corresponds to $\theta_0 = 1$.

E-bank: Let

$$v(\theta) = F(\kappa_E)\,p_H(R - R_o) + (1 - F(\kappa_E))\,B \tag{47}$$

be the borrower's expected payoff if the asset matures. The borrower's expected payoff at $t=0$ is

$$V_E(\theta_0) = \int_0^\infty e^{-(\rho+\phi)t}\phi v(\theta_t)\,dt = \Phi B + \frac{\phi(b-B)}{\bar\kappa}\frac{\Delta R_o \theta_0}{2\rho+\phi}, \tag{48}$$

where we have substituted $R_o = R - \frac{b}{\Delta}$ and therefore $p_H(R-R_o) = b$. Clearly, $V_E(\theta_0) + \Pi_E(\theta_0)$ is maximized at

$$\theta_0 = 1 - \frac{\rho+\phi}{2\rho+\phi}\frac{B-b}{\Delta R_o}. \tag{49}$$

S-bank: If the debt issuance constraint is binding, $\Pi_S(\theta)$ solves

$$\frac{\Delta^2}{2\bar\kappa}\left[(R_o\theta)^2 - D^2\right] = D.$$

The previous equation has two roots, and the positive root is

$$\Pi_S(\theta) = -\frac{\bar\kappa}{\Delta^2} + \sqrt{\left(\frac{\bar\kappa}{\Delta^2}\right)^2 + (R_o\theta)^2}.$$

1. If $\Phi \geq \underline{\Phi} := \sqrt{\frac{(\bar\kappa/\Delta^2)^2}{(\bar\kappa/\Delta^2)^2+(R_o)^2}}$, the bank's debt choice satisfies $D^*(1) = \tilde D(1) = \frac{\rho\bar\kappa}{\phi\Delta^2}$.
2. Otherwise, the bank's debt choice satisfies $D^*(1) = \Pi_S(1) = -\frac{\bar\kappa}{\Delta^2} + \sqrt{\left(\frac{\bar\kappa}{\Delta^2}\right)^2 + (R_o)^2}$.

The borrowing capacity is $\Pi_S(1) = \Phi\pi_S(1)$ and the optimal debt when $\theta = 1$ is

$$D^*(1) = \min\left\{\frac{\rho\bar\kappa}{\phi\Delta^2}, -\frac{\bar\kappa}{\Delta^2} + \sqrt{\left(\frac{\bar\kappa}{\Delta^2}\right)^2 + (R_o)^2}\right\}.$$

The debt-issuance constraint is slack if

$$\tilde D(1) = \frac{\rho\bar\kappa^2}{\phi\Delta^2} \leq \Pi_S(1) = -\frac{\bar\kappa}{\Delta^2} + \sqrt{\left(\frac{\bar\kappa}{\Delta^2}\right)^2 + R_o^2}.$$

Simple derivation shows this is satisfied if and only if

$$\Phi > \sqrt{\frac{(\bar\kappa/\Delta^2)^2}{R_o^2 + (\bar\kappa/\Delta^2)^2}},$$

which holds if and only if ρ is sufficiently low.

Whenever the borrowing constraint is slack (that is $\tilde D(1) < \Pi_S(1)$), we can plug in $D^*(1)$ to get $\pi_S(1) = \frac{\Delta^2}{2\bar\kappa}\left(R_o - \frac{\rho\bar\kappa}{\phi\Delta^2}\right)^2 + \frac{\rho + \phi\frac{\Delta^2}{\bar\kappa}\left(R_o - \frac{\rho\bar\kappa}{\phi\Delta^2}\right)}{\phi}\frac{\rho\bar\kappa}{\phi\Delta^2}$.

According to (4), $\kappa_S = \Delta\left(R_o - \frac{\rho\bar\kappa}{\phi\Delta^2}\right)$ and $p_S(1) = \frac{\kappa_S}{\bar\kappa}\Delta = \frac{\Delta^2}{\bar\kappa}\left(R_o - \frac{\rho\bar\kappa}{\phi\Delta^2}\right)$. Consequently, $\hat\pi_S(1, D^*)$

and $\pi_S(1)$ defined in (17) and (21) become

$$\hat{\pi}_S(1, D^*) = \frac{\Delta^2}{2\bar{\kappa}}\left(R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2}\right)^2 \tag{50}$$

$$\pi_S(1) = \frac{\Delta^2}{2\bar{\kappa}}\left(R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2}\right)^2 + \frac{\rho + \phi\frac{\Delta^2}{\bar{\kappa}}\left(R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2}\right)}{\phi}\frac{\rho\bar{\kappa}}{\phi\Delta^2}. \tag{51}$$

Again, we can define the entrepreneur and bank's payoff as

$$v(1) = F(\kappa_S)\, p_H (R - R_o) + (1 - F(\kappa_S)) B = B + \frac{\Delta\left(R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2}\right)}{\bar{\kappa}}(b - B) \tag{52}$$

$$V_S(1) = \int_0^\infty e^{-(\rho+\phi)t}\phi v(\theta_t)\, dt = \Phi v(1) \tag{53}$$

$$\Pi_S(1) = \Phi\pi_S(1). \tag{54}$$

Proof of Proposition 3

Proof. Simple calculations show that

$$W_E(1) = \frac{\phi}{\rho+\phi}\left[\frac{(\Delta R_o)^2}{2\bar{\kappa}} - (B-b)\frac{\rho+\phi}{2\rho+\phi}\frac{\Delta R_o}{\bar{\kappa}} + B\right]$$

$$W_S(1) = -\frac{\Phi\Delta^2}{2\bar{\kappa}}(D^*(1))^2 + \left[1 - \Phi + \Phi(B-b)\frac{\Delta}{\bar{\kappa}}\right] D^*(1) + \Phi\left[\frac{(R_o\Delta)^2}{2\bar{\kappa}} + B - (B-b)\frac{R_o\Delta}{\bar{\kappa}}\right].$$

The optimal debt issuance is

$$D^*(1) = \min\left\{\frac{\rho\bar{\kappa}}{\phi\Delta^2}, \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o)^2} - \frac{\bar{\kappa}}{\Delta^2}\right\}.$$

In the slack case,

$$D^*(1) = \frac{\rho\bar{\kappa}}{\phi\Delta^2} \leq \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o)^2} - \frac{\bar{\kappa}}{\Delta^2} \Rightarrow \Phi \geq \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 \Big/ \left((R_o)^2 + \left(\frac{\bar{\kappa}}{\Delta^2}\right)^2\right)}.$$

We have

$$\begin{aligned}
\Delta W &= \frac{\rho}{\rho+\phi}\left[\frac{(B-b)}{\Delta}\left(\frac{\phi}{2\rho+\phi}\frac{R_o\Delta^2}{\bar{\kappa}}-1\right)-\frac{\rho\bar{\kappa}}{2\phi\Delta^2}\right] \\
&\le \frac{\rho}{\rho+\phi}\left[\frac{(B-b)}{\Delta}\left(\frac{\phi}{2\rho+\phi}\left(\frac{\rho}{\phi}+\Delta\right)-1\right)-\frac{\rho\bar{\kappa}}{2\phi\Delta^2}\right] \\
&= -\frac{\rho}{\rho+\phi}\left[\frac{(B-b)}{\Delta}\frac{\rho+\phi(1-\Delta)}{2\rho+\phi}+\frac{\rho\bar{\kappa}}{2\phi\Delta^2}\right] \\
&< 0.
\end{aligned}$$

□

Detailed Analysis of subsection 5.3

After $t=0$, the bank only chooses its trading strategy and solves

$$\Pi_P(\theta_0, D) = \max_{\{\theta_t\}_{t>0}} \mathbb{E}\left[\int_0^\infty e^{-(\rho+\phi)t}\left(\phi\pi_P(\theta_t)\,dt + dG(\theta_t)\right)\right].$$

The entrepreneur's expected payoff is

$$\begin{aligned}
V_P(\theta_0, D) &= \mathbb{E}\left[\int_0^\infty e^{-(\rho+\phi)t}\phi\left\{\mathbf{1}_{\{\kappa\le\kappa_P\}}p_H R_f + \mathbf{1}_{\{\kappa>\kappa_P\}}\left(p_L R_f + B\right)\right\}dt\right] \\
&= \int_0^\infty e^{-(\rho+\phi)t}\phi\left[p_L R_f + B - (B-b)F(\kappa_P)\right]dt
\end{aligned}$$

The initial surplus is

$$\begin{aligned}
W_P(\theta_0, D) &= V_P(\theta_0, D) + (1-\theta_0)q_P(\theta_0, D) + \Pi_P(\theta_0, D) + Dq_D(\theta_0, D) \\
&= V_P(\theta_0, D) + \Pi_P(\theta_0, D) + \left(1-\theta_0+\frac{D}{R_o}\right)q_P(\theta_0, D).
\end{aligned}$$

The rest of the analysis follows from subsection 4.1. We can get

$$\dot{\theta} = -\frac{\phi(1-\Phi)\pi_P'(\theta)}{\Phi\pi_P''(\theta)} = -\frac{\phi(1-\Phi)R_o(p_L+F(\kappa_P)\Delta)}{\Phi(\Delta R_o)^2 f(\kappa_P)} < 0.$$

In the linear-quadratic example, we have

$$\begin{aligned}
W_P(\theta_0, D) &= V_P(\theta_0, D) + (1-\theta_0)q_P(\theta_0, D) + \Pi_P(\theta_0, D) + Dq_D(\theta_0, D) \\
&= -\frac{\Phi\Delta^2}{2\bar{\kappa}}D^2 + \frac{\Phi\Delta}{\bar{\kappa}}\left[\frac{B-b}{2-\Phi} - \Delta R_o(1-\theta_0)\right]D \\
&\quad + \Phi\left[-\frac{(\Delta R_o)^2}{2\bar{\kappa}}(\theta_0)^2 + \frac{\Delta R_o}{\bar{\kappa}}\left(\Delta R_o - \frac{B-b}{2-\Phi}\right)\theta_0 + B\right].
\end{aligned}$$

From here, we have

$$D^*(\theta_0) = \frac{B-b}{\Delta(2-\Phi)} - R_o(1-\theta_0).$$

If we plug $D^*(\theta_0)$ back to $W_P(\theta_0, D)$, we get

$$W_P(\theta_0, D^*(\theta_0)) = \Phi\left[\frac{B-b}{\bar{\kappa}(2-\Phi)}\left(\frac{B-b}{2(2-\Phi)} - \Delta R_o\right) + \frac{(\Delta R_o)^2}{2\bar{\kappa}} + B\right],$$

which is independent of θ_0. This proves that there is a set of combinations between D and θ_0 that maximize the initial surplus so that the solution is indeterminate. In the case of $\theta_0 = 1$, we have

$$D^* = \frac{B-b}{\Delta(2-\Phi)}.$$

E Detailed Analysis of Subsection 6.1

E.1 Lender Dispersion

First, we prove that $L_E(1) = \max_{\theta_0} L_E(\theta_0)$, so that in the E-bank equilibrium, $\theta_0 = 1$ maximizes the entrepreneur's initial borrowing. Since $q_E(\theta_0) = \Phi\pi'_E(\theta_0)$, we have

$$\begin{aligned}
&L_E(\theta_0) = \Phi\pi_E(\theta_0) + (1-\theta_0) \\
&L'_E(\theta_0) = \Phi\pi'_E(\theta_0) - \Phi\pi'_E(\theta_0) + (1-\theta_0)\,\Phi\pi''_E(\theta_0) = (1-\theta_0)\,\Phi\pi''_E(\theta_0).
\end{aligned}$$

From

$$\pi'_E(\theta_0) = d_E(\theta_0) = R_o p_E(\theta_0) = R_o\left(p_L + F(\kappa_E)\,\Delta\right) = R_o\left(p_L + F(\Delta R_o\theta_0)\,\Delta\right),$$

we have

$$\pi''_E(\theta_0) = d'_E(\theta_0) = f(\Delta R_o\theta_0)\,(\Delta R_o)^2.$$

Therefore,

$$L'_E(\theta_0) = (1-\theta_0)\,\Phi d'_E(\theta_0) = (1-\theta_0)\,\Phi f(\Delta R_o\theta_0)\,(\Delta R_o)^2 \geq 0.$$

Next, we explore lender dispersion in the S-bank equilibrium in the linear-quadratic example. We will show that depending on parameters, the optimal θ_0 can be either interior or 1. Note that if $p_L = 0$, then $\theta_\dagger = 0$. Therefore, for any $\theta_0 \in (0,1)$, the bank buys loans back smoothly. Moreover, it is straightforward to get

$$\begin{aligned}
\hat{p}_S(\theta, D) &= \frac{\Delta^2\left(R_o\theta - D\right)}{\bar{\kappa}} \\
\hat{\pi}_S(\theta, D) &= \frac{\Delta^2\left(R_o\theta - D\right)}{\bar{\kappa}}\left(R_o\theta - D\right) - \frac{\kappa_S^2}{2\bar{\kappa}} = \frac{\left(\Delta\left(R_o\theta - D\right)\right)^2}{2\bar{\kappa}} \\
y_t = \hat{y}(\theta, D) &= \phi\left(1 - \hat{p}_S(\theta, D)\right) = \frac{\phi}{\bar{\kappa}}\left(\bar{\kappa} - \Delta^2\left(R_o\theta - D\right)\right) = \phi - \frac{\phi}{\bar{\kappa}}\Delta^2\left(R_o\theta - D\right) \\
\mathcal{V}(D, \theta) &= \Phi\pi_S(\theta) = \Phi\left[\frac{\Delta^2\left(R_o\theta - D\right)}{\bar{\kappa}}\theta R_o - \frac{\left(\Delta\left(R_o\theta - D\right)\right)^2}{2\bar{\kappa}}\right] + (1-\Phi)D \\
&= -\Phi\frac{\Delta^2}{2\bar{\kappa}}D^2 + (1-\Phi)D + \Phi\frac{\left(\Delta R_o\theta\right)^2}{2\bar{\kappa}} \\
\mathcal{L} &= \mathcal{V}(D, \theta) + \Phi z(\theta)\left[\pi_S(\theta) - D\right] \\
\mathcal{L}_D &= -\Phi\frac{\Delta^2}{\bar{\kappa}}D + (1-\Phi) - \Phi z(\theta) = 0 \\
z(\theta) &= \frac{\rho}{\phi} - \frac{\Delta^2}{\bar{\kappa}}D^*(\theta)
\end{aligned}$$

From here, we can derive the optimal short-term debt

$$D^*(\theta) = \min\left\{\frac{\rho\bar{\kappa}}{\phi\Delta^2}, \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + \left(R_o\theta\right)^2} - \frac{\bar{\kappa}}{\Delta^2}\right\},$$

and the threshold in Lemma 2 is

$$\theta_D = \bar{\kappa}\frac{\sqrt{1-\Phi^2}}{\Phi R_o\Delta^2}.$$

Below, we separately analyze the slack and the binding region.

Slack region. In this case, we have $z(\theta) = 0$ and

$$D^* = \frac{\rho\bar{\kappa}}{\phi\Delta^2}.$$

Simple calculations get

$$\Pi_S(\theta) = \mathcal{V}\left(D^*, \theta\right) = \Phi\pi_S(\theta) = \Phi\left(\frac{\left(\Delta R_o\theta\right)^2}{2\bar{\kappa}} + \frac{\rho}{2\phi}\frac{\rho\bar{\kappa}}{\phi\Delta^2}\right).$$

Moreover, we get

$$
\begin{aligned}
&{\Pi_S}'(\theta) = \Phi \frac{(\Delta R_o)^2}{\bar{\kappa}} \theta, \qquad \pi_S(\theta) = \frac{\rho}{2\phi} \frac{\rho \bar{\kappa}}{\phi \Delta^2} + \frac{(\Delta R_o \theta)^2}{2\bar{\kappa}} \\
&{\pi_S}'(\theta) = \frac{(\Delta R_o)^2}{\bar{\kappa}} \theta, \qquad {\pi_S}''(\theta) = \frac{(\Delta R_o)^2}{\bar{\kappa}} \\
&q_S(\theta) = \Phi \frac{(\Delta R_o)^2}{\bar{\kappa}} \theta, \qquad \kappa_S = \Delta \left(R_o \theta - D^*\right) = \Delta \left(R_o \theta - \frac{\rho \bar{\kappa}}{\phi \Delta^2}\right) \\
&\frac{\partial \kappa_S}{\partial \theta} = \Delta R_o, \qquad F\left(\kappa_S\right) = \frac{\Delta}{\bar{\kappa}} \left(R_o \theta - \frac{\rho \bar{\kappa}}{\phi \Delta^2}\right) \\
&p_S(\theta) = \hat{p}_S\left(\theta, D^*\right) = \frac{\Delta^2}{\bar{\kappa}} \left(R_o \theta - \frac{\rho \bar{\kappa}}{\phi \Delta^2}\right), \qquad d_S(\theta) = \frac{R_o \Delta^2}{\bar{\kappa}} \left(R_o \theta - \frac{\rho \bar{\kappa}}{\phi \Delta^2}\right).
\end{aligned}
$$

Note that under $\bar{\kappa} \geq \Phi R_o \Delta^2$, we have $p_S(\theta) \in [0,1]$. From the evolution of θ_t, we get

$$
\theta_t = \left(\theta_0 - \frac{\bar{\kappa}}{\Phi R_o \Delta^2}\right) e^{-\rho t} + \frac{\bar{\kappa}}{\Phi R_o \Delta^2},
$$

for $t < T_1$, where $T_1 = \frac{1}{\rho} \ln \frac{\Phi R_o \Delta^2 \theta_0 - \bar{\kappa}}{\Phi R_o \Delta^2 - \bar{\kappa}}$ satisfies

$$
\theta_{T_1} = \left(\theta_0 - \frac{\bar{\kappa}}{\Phi R_o \Delta^2}\right) e^{-\rho T_1} + \frac{\bar{\kappa}}{\Phi R_o \Delta^2} = 1.
$$

All together, we have

$$
\theta_t = \begin{cases} \left(\theta_0 - \frac{\bar{\kappa}}{\Phi R_o \Delta^2}\right) e^{-\rho t} + \frac{\bar{\kappa}}{\Phi R_o \Delta^2}, & 0 \leq t < T_1 \\ 1 & t \geq T_1. \end{cases}
$$

From here, we get

$$
\begin{aligned}
&L_S\left(\theta_0\right) = \Phi \left(-\frac{(\Delta R_o)^2}{2\bar{\kappa}} \theta_0^2 + \frac{(\Delta R_o)^2}{\bar{\kappa}} \theta_0 + \frac{\rho}{2\phi} \frac{\rho \bar{\kappa}}{\phi \Delta^2}\right) \\
&L_S\left(\theta_D\right) = R_o \sqrt{1 - \Phi^2} - \frac{\bar{\kappa}}{\Delta^2} \frac{\rho}{\rho + \phi}.
\end{aligned}
$$

Let

$$
v(\theta) = p_H R_f F\left(\kappa_S\right) + B\left(1 - F\left(\kappa_S\right)\right) = (b - B) F\left(\kappa_S\right) + B = (b - B) \frac{\Delta}{\bar{\kappa}} \left(R_o \theta - \frac{\rho \bar{\kappa}}{\phi \Delta^2}\right) + B,
$$

we have

$$\begin{aligned}V_S(\theta_0) &= \int_0^\infty \phi e^{-(\rho+\phi)t} v(\theta_t)\,dt = \int_0^\infty \phi e^{-(\rho+\phi)t}\left[(b-B)F(\kappa_S)+B\right]dt \\ &= \phi\frac{\Delta}{\bar\kappa}(b-B)R_o\left[\int_0^{T_1} e^{-(\rho+\phi)t}\theta_t dt + \int_{T_1}^\infty e^{-(\rho+\phi)t}dt\right] + \Phi\left(B - \frac{\rho(b-B)}{\phi\Delta}\right).\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}V_S(\theta_0) = \phi\frac{\Delta}{\bar\kappa}(b-B)R_o&\left[\frac{\theta_0}{2\rho+\phi} + \left(\frac{\bar\kappa-\Phi R_o\Delta^2\theta_0}{\bar\kappa-\Phi R_o\Delta^2}\right)^{-\frac{\rho+\phi}{\rho}}\frac{\rho}{\rho+\phi}\frac{1}{2\rho+\phi}\left(1-\frac{\bar\kappa}{\Phi R_o\Delta^2}\right) + \frac{\bar\kappa}{\Phi R_o\Delta^2}\frac{\rho}{\rho+\phi}\frac{1}{2\rho+\phi}\right] \\ &+ \Phi\left(B-\frac{\rho(b-B)}{\phi\Delta}\right),\end{aligned}$$

and

$$\begin{aligned}V_S(\theta_D) = \frac{1}{\Delta}\frac{(b-B)}{2\rho+\phi}&\left[\rho + (\rho+\phi)\sqrt{1-\Phi^2} + \left(\Phi R_o\Delta^2-\bar\kappa\right)\frac{\rho}{\bar\kappa}\left(\frac{\bar\kappa-\bar\kappa\sqrt{1-\Phi^2}}{\bar\kappa-\Phi R_o\Delta^2}\right)^{-\frac{\rho+\phi}{\rho}}\right] \\ &+ \Phi\left(B-\frac{\rho(b-B)}{\phi\Delta}\right).\end{aligned}$$

Finally, in this region we have

$$\begin{aligned}W_S(\theta_0) =& L_S(\theta_0) + V_S(\theta_0) \\ =& -\Phi\frac{(\Delta R_o)^2}{2\bar\kappa}\theta_0^2 + \frac{\Delta R_o}{\bar\kappa}\left(\Phi\Delta R_o + \phi\frac{(b-B)}{2\rho+\phi}\right)\theta_0 + \left(\frac{\bar\kappa-\Phi R_o\Delta^2\theta_0}{\bar\kappa-\Phi R_o\Delta^2}\right)^{-\frac{\rho+\phi}{\rho}}\frac{\rho(b-B)}{2\rho+\phi}\left(\frac{\Phi\Delta R_o}{\bar\kappa}-\frac{1}{\Delta}\right) \\ &+ \frac{\rho}{\Delta}\frac{(b-B)}{2\rho+\phi} + \Phi\left(B - \frac{\rho(b-B)}{\phi\Delta} + \frac{\rho}{2\phi}\frac{\rho\bar\kappa}{\phi\Delta^2}\right),\end{aligned}$$

and

$$\begin{aligned}W_S'(\theta_0) &= \frac{\Phi(\Delta R_o)^2}{\bar\kappa}(1-\theta_0) - \frac{\phi\Delta R_o}{\bar\kappa}\frac{B-b}{2\rho+\phi}\left[1-\left(\frac{\bar\kappa-\Phi R_o\Delta^2\theta_0}{\bar\kappa-\Phi R_o\Delta^2}\right)^{-\frac{2\rho+\phi}{\rho}}\right], \\ W_S''(\theta_0) &= \frac{\phi\Delta R_o}{\bar\kappa}\frac{\Phi R_o\Delta^2}{\bar\kappa-\Phi R_o\Delta^2}\frac{B-b}{\rho}\left(\frac{\bar\kappa-\Phi R_o\Delta^2\theta_0}{\bar\kappa-\Phi R_o\Delta^2}\right)^{-\frac{2\rho+\phi}{\rho}-1} - \frac{\Phi(\Delta R_o)^2}{\bar\kappa}.\end{aligned}$$

Binding region. In this region, $D^*(\theta) = \Pi_S(\theta)$ and $\Pi_S(\theta) = \Phi\pi_S(\theta) = \mathcal{V}(\Pi_S(\theta),\theta)$. we have

$$\begin{aligned}\Pi_S(\theta) &= D^*(\theta) = \sqrt{\left(\frac{\bar\kappa}{\Delta^2}\right)^2 + (R_o\theta)^2} - \frac{\bar\kappa}{\Delta^2} \\ z(\theta) &= \frac{\rho}{\phi} - \frac{\Delta^2}{\bar\kappa}D^*(\theta) = \frac{1}{\Phi} - \frac{1}{\bar\kappa}\sqrt{(\bar\kappa)^2 + (\Delta^2 R_o\theta)^2}.\end{aligned}$$

Clearly, $z(\theta_D) = z\left(\bar{\kappa}\frac{\sqrt{1-\Phi^2}}{\Phi R_o \Delta^2}\right) = 0$. Since $z(\theta)$ is decreasing in θ, we verify that $z(\theta) \geq 0$ over $\theta \in [0, \theta_D]$. Also $z(\theta)$ is continuous over $\theta \in [0,1]$. Simple calculations show that

$$\kappa_S = \Delta\left[R_o\theta + \frac{\bar{\kappa}}{\Delta^2} - \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o\theta)^2}\right], \qquad p_S(\theta) = \frac{1}{\bar{\kappa}}\left(\Delta^2 R_o\theta + \bar{\kappa} - \sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta)^2}\right)$$

$$d_S(\theta) = \frac{R_o}{\bar{\kappa}}\left(\Delta^2 R_o\theta + \bar{\kappa} - \sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta)^2}\right), \qquad \Pi_S'(\theta) = \frac{(\Delta R_o)^2\theta}{\sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta)^2}}.$$

Moreover, we have

$$L_S(\theta_0) = \Pi_S(\theta_0) + q_S(\theta_0)(1-\theta_0) = \frac{1}{\Delta^2}\sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta_0)^2} + \frac{(\Delta R_o)^2\theta_0(1-\theta_0)}{\sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta_0)^2}} - \frac{\bar{\kappa}}{\Delta^2}.$$

From

$$v(\theta) = p_H R_f F(\kappa_S) + B(1 - F(\kappa_S)) = (b-B)F(\kappa_S) + B,$$

we have

$$V_S(\theta_0) = \int_0^\infty \phi e^{-(\rho+\phi)t} v(\theta_t)\,dt = \int_0^{T_D} \phi e^{-(\rho+\phi)t} v(\theta_t)\,dt + \int_{T_D}^{T_1} \phi e^{-(\rho+\phi)t} v(\theta_t)\,dt + \int_{T_1}^\infty \phi e^{-(\rho+\phi)t} v(\theta_t)\,dt,$$

where T_D is the first time θ_t hits θ_D from below and T_1 is the first time θ_t hits 1 from below. When $t \in [0, T_D]$,

$$\begin{aligned}
\frac{d\theta_t}{dt} &= -\phi\frac{d_S(\theta_t) - q_S(\theta_t)}{q_S{}'(\theta_t)} \\
&= \frac{\phi}{\bar{\kappa}}\frac{\left(\bar{\kappa}^2 + (\Delta^2 R_o\theta_t)^2\right)}{(\Delta R_o\bar{\kappa})^2}\sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta_t)^2}\left(\sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta_t)^2} - \bar{\kappa}\right)\left[R_o - \frac{(\Delta R_o)^2\theta_t}{\sqrt{\bar{\kappa}^2 + (\Delta^2 R_o\theta_t)^2}}\right],
\end{aligned}$$

and

$$v(\theta_t) = \frac{\Delta R_o}{\bar{\kappa}}(b-B)\theta_t - (b-B)\frac{\Delta}{\bar{\kappa}}\sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o\theta_t)^2} + \left(\frac{b-B}{\Delta} + B\right).$$

When $t \in [T_D, T_1]$,

$$\theta_t = \begin{cases} \frac{\bar{\kappa}}{\Phi R_o\Delta^2}\left[\left(\sqrt{1-\Phi^2} - 1\right)e^{-\rho(t-T_D)} + 1\right], & T_D \leq t < T_1 \\ 1 & t \geq T_1. \end{cases}$$

Let

$$\theta_{T_1} = 1 \Rightarrow T_1 = e^{-(\rho+\phi)\left[\frac{1}{\rho}\ln\frac{\bar{\kappa}\left(1-\sqrt{1-\Phi^2}\right)}{\bar{\kappa}-\Phi R_o\Delta^2} + T_D\right]}.$$

Then

$$\begin{aligned}\int_{T_D}^{T_1} \phi e^{-(\rho+\phi)t} v(\theta_t)\, dt =&\phi(b-B)\frac{\Delta R_o}{\bar{\kappa}} \int_{T_D}^{T_1} e^{-(\rho+\phi)t}\theta_t dt + \phi\left(B - \frac{\rho(b-B)}{\phi\Delta}\right)\int_{T_D}^{T_1} e^{-(\rho+\phi)t} dt \\ =&\frac{\Delta R_o}{\bar{\kappa}}\frac{\phi(b-B)}{2\rho+\phi}\left[\left(\frac{\rho\bar{\kappa}}{\phi R_o \Delta^2}+\theta_D\right)e^{-(\rho+\phi)T_D} - \left(\frac{\rho\bar{\kappa}}{\phi R_o\Delta^2}+1\right)e^{-(\rho+\phi)T_1}\right] \\ &+\Phi\left(B-\frac{\rho(b-B)}{\phi\Delta}\right)\left(e^{-(\rho+\phi)T_D} - e^{-(\rho+\phi)T_1}\right)\end{aligned}$$

When $t \in [T_1, \infty]$,

$$\int_{T_1}^{\infty} \phi e^{-(\rho+\phi)t} v(\theta_t)\, dt = \Phi e^{-(\rho+\phi)\left[\frac{1}{\rho}\ln\frac{\bar{\kappa}\left(1-\sqrt{1-\Phi^2}\right)}{\bar{\kappa}-\Phi R_o\Delta^2}+T_D\right]}\left(\frac{\Delta R_o}{\bar{\kappa}}(b-B) + B - \frac{\rho(b-B)}{\phi\Delta}\right).$$

Optimal θ_0^*. We show that the optimal θ_0^* may be interior or 1. In the slack region, we have

$$\begin{aligned}&W_S'(1) = 0, \\ &W_S''(1) = \frac{\Phi(\Delta R_o)^2}{\bar{\kappa}}\left[\frac{\phi\Delta}{\bar{\kappa}-\Phi R_o\Delta^2}\frac{B-b}{\rho} - 1\right].\end{aligned}$$

The sign of $W_S''(1)$ depends on the choices of parameters. If $B-b$ is small, $W_S''(1) < 0$ maybe negative and as a result $\theta_0 = 1$ can be optimal. If $B-b$ is large, $W_S''(1) > 0$ and the optimal $\theta_0 \in (0,1)$.

Moreover, the optimal θ_0^* could be in either the binding or the slack region. When

$$W_S'(\theta_D) = \frac{\Phi(\Delta R_o)^2}{\bar{\kappa}} - R_o\sqrt{1-\Phi^2} - \frac{\phi\Delta R_o}{\bar{\kappa}}\frac{B-b}{2\rho+\phi}\left[1-\left(\frac{\bar{\kappa}-\bar{\kappa}\sqrt{1-\Phi^2}}{\bar{\kappa}-\Phi R_o\Delta^2}\right)^{-\frac{2\rho+\phi}{\rho}}\right] > 0,$$

the optimal initial θ_0^* lies in the slack region. Otherwise, it can be in the constrained region.

E.2 Short-term debt pricing

We offer the Proof of Proposition 4.

Proof. If θ is sufficiently large such that the debt issuance constraint is slack, simple derivation shows that the optimal debt issuance satisfies

$$\tilde{D}(\theta) = \frac{(1-\Phi) - \Phi(1-p(\theta))(1-\xi)}{\Phi[1-(1-p(\theta))(1-\xi)]\, f(\kappa_S)\,\Delta^2}.$$

In the region where the debt-issuance constraint is slack, the HJB equation implies

$$\dot{\theta} = \phi\frac{R_o\xi\frac{(1-\Phi)-\Phi(1-p(\theta))(1-\xi)}{1-(1-p(\theta))(1-\xi)} - (1-\Phi)\,p(\theta)R_o}{\Phi\pi_S''(\theta)}.$$

$\square$

E.3 Continuous Monitoring Technology

Suppose the project generates a payoff of R at maturity with a probability of 1 unless it fails before that maturity event arrives. The failure probability depends on the bank's monitoring effort: before maturity, the project fails with an arrival rate $\eta(1-m_t)$, where m_t is the bank's monitoring effort chosen at time t. The cost of monitoring is given by a convex function $h(m_t)$. Let $m_t = m$.

E-bank By the envelope theorem, we have

$$(\rho + \eta(1-m(\theta)))\, \Pi'_E(\theta) = \phi\left[R_o - \Pi'_E(\theta)\right] + \dot{\theta}\left[\Pi''_E(\theta) - q'_E(\theta)\right]$$

Substituting $\Pi'_E(\theta) = q_E(\theta)$ and $\Pi''_E(\theta) = q'_E(\theta)$ we get

$$(\rho + \eta(1-m(\theta)))\, q_E(\theta) = \phi\left[R_o - q_E(\theta)\right]$$

It follows that

$$\rho q_E(\theta) = -\dot{\theta} q'_E(\theta) \Longrightarrow \dot{\theta} = -\rho \frac{q_E(\theta)}{q'_E(\theta)} = -\rho \frac{\Pi'_E(\theta)}{\Pi''_E(\theta)}$$

In the particular case that $h(m) = km^2/2$ we have $(h')^{-1}(x) = x/k$, so

$$m(\theta) = \frac{\eta}{k}\Pi_E(\theta).$$

For simplicity, we will consider parameters such that $\frac{\eta}{k}\Pi_E(\theta) < 1$, so the solution is interior. Substituting in the HJB equation, and defining $\upsilon \equiv \eta^2/k$ we get

$$\frac{\upsilon}{2}\Pi_E(\theta)^2 - (\rho + \phi + \eta)\,\Pi_E(\theta) + \phi R_o \theta = 0.$$

The solution must be increasing in θ. Hence the relevant root is

$$\Pi_E(\theta) = \frac{(\rho + \phi + \eta) - \sqrt{(\rho + \phi + \eta)^2 - 2\upsilon\phi R_o \theta}}{\upsilon}.$$

The derivatives for the value function are

$$\Pi'_E(\theta) = \phi R_o \left[(\rho + \phi + \eta)^2 - 2\upsilon\phi R_o\theta\right]^{-1/2} > 0$$

$$\Pi''_E(\theta) = \upsilon(\phi R_o)^2 \left[(\rho + \phi + \eta)^2 - 2\upsilon\phi R_o\theta\right]^{-3/2} > 0$$

The optimal monitoring is

$$m(\theta) = \frac{(\rho+\phi+\eta) - \sqrt{(\rho+\phi+\eta)^2 - 2\upsilon\phi R_o\theta}}{\eta}.$$

Notice that the condition $\frac{\eta}{k}\Pi_E(\theta) < 1$ is satisfied for all $\theta \in [0,1]$ if

$$(\rho+\phi+\eta) - \sqrt{(\rho+\phi+\eta)^2 - 2\upsilon\phi R_o} < \upsilon\frac{k}{\eta} = \eta \Longleftrightarrow R_o < \frac{k\left(\rho+\phi+\frac{\eta}{2}\right)}{\eta\phi}.$$

In equilibrium, trading satisfies

$$\dot{\theta} = -\rho\frac{(\rho+\phi+\eta)^2 - 2\upsilon\phi R_o\theta}{\upsilon\phi R_o} < 0.$$

Finally, note that

$$\Pi_E'(0) = \frac{\phi R_o}{\rho+\phi+\eta} < q_E(0) = \frac{\phi R_o}{\phi+\eta},$$

which implies that the bank has incentives to sell an atom for θ sufficiently low. As in the benchmark model, when the $\theta < \theta^*$, where θ_* satisfies $\Pi_E'(0) = q_E(0)$, the bank liquidates the remaining loans immediately. At θ_*, the bank follows a mixed strategy like the one in the benchmark model.

S-bank Substituting $\phi\pi_S(\theta) = \phi\theta R_o + \rho\Pi_S(\theta)$, we get

$$\eta\Pi_S(\theta) = \max_m \phi\Big[\theta R_o - \Pi_S(\theta)\Big] + \eta m\Pi_S(\theta) - h(m).$$

In the case that $h(m) = km^2/2$, the HJB equation reduces to

$$\frac{\upsilon}{2}\Pi_S(\theta)^2 - (\eta+\phi)\Pi_S(\theta) + \phi\theta R_o = 0,$$

so we get

$$\Pi_S(\theta) = \frac{(\phi+\eta) - \sqrt{(\phi+\eta)^2 - 2\upsilon\phi R_o\theta}}{\upsilon}$$

and

$$m(\theta) = \frac{(\phi+\eta) - \sqrt{(\phi+\eta)^2 - 2\upsilon\phi R_o\theta}}{\eta}$$

Using the envelope theorem, we get that

$$\eta(1-m(\theta))\Pi_S'(\theta) = \phi\Big[R_o - \Pi_S'(\theta)\Big].$$

From the loan pricing equation,

$$\eta(1-m)q_S(\theta) = \phi\left(R_o - q_S(\theta)\right) + \dot{\theta}q_S(\theta),$$

we plug in $q_S(\theta) = \Pi'_S(\theta)$ and $q'_S(\theta) = \Pi''_S(\theta)$ to get that

$$\dot{\theta} = 0.$$

Notice that

$$(\phi + \eta) - \sqrt{(\phi + \eta)^2 - 2\upsilon\phi R_o\theta} > (\rho + \phi + \eta) - \sqrt{(\rho + \phi + \eta)^2 - 2\upsilon\phi R_o\theta}$$

so $\Pi_S(\theta) > \Pi_E(\theta)$ and $m_S(\theta) > m_E(\theta)$.

E.4 S-bank with Loss-Absorbing Equity and Riskless Debt

Following the solution, we have the bank's value function under $\theta_t = \theta$ is

$$\Pi_S(\theta) = \Phi\pi_S(\theta) = \Phi\left[\hat{p}_S(\theta, D)(\theta R_o - D) + (1 - \hat{p}_S(\theta, D))(-X) - \int_0^{\kappa_S} \kappa dF(\kappa)\right].$$

At $t = 0$, the bank chooses $\{\theta_0, D\}$ to maximize

$$\Pi_S(\theta_0, D) + q_D(\theta_0, D)D,$$

where

$$q_D(\theta_0, D) = \mathbb{E}\left[p_S(\theta) + (1 - p_S(\theta))\frac{X}{D}\right]$$

is the price of long-term debt at $t = 0$, and the expectation operator takes into account the trajectory of θ_t after $\theta_0 = \theta$ and the shock for project maturing.