# Debt Maturity Management

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**Abstract**

How does a borrower choose between long- and short-term debt when she has no commitment to future issuance policy? Short-term debt protects creditors from future dilution; long-term debt allows the borrower to share losses with creditors in a downturn by effectively reducing her payments. We develop a theory of debt maturity that highlights the tradeoff between commitment and risk management. Borrowers far from default value risk management and use a combination of long- and short-term debt. By contrast, distressed borrowers exclusively issue short-term debt. Our model predicts pro-cyclical leverage, debt maturity, and long-term debt issuance.

**Keywords:** capital structure; debt maturity; risk management; dynamic tradeoff theory.

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# 1 Introduction

The optimal management of debt obligations is a central problem faced by indebted entities, including households, firms, and sovereign governments. In practice, debt can differ in a number of aspects, and an important one is its maturity. Borrowing can be short as in the case of repurchase agreement and trade credit, or long as in the case of 30-year corporate bonds. How do borrowers choose the maturity profile of their outstanding debt? How do they adjust the mix between long- and short-term borrowing, following shocks to their enterprise value?

Yet, the academic literature falls behind in providing a useful framework to study these questions, despite the obvious importance. For example, the Leland model (Leland, 1994) and the vast follow-up literature typically assume 1) all debt has the same (expected) maturity, and 2) the borrower either commits to the total leverage or may increase leverage only after paying some exogenous issuance cost.[1] While these assumptions simplify the analysis, they are not consistent with the ample empirical evidence that borrowers often times issue a mix of long- and short-term debt, and the adjustment of debt maturity structure can be slow and take time to accomplish.

In this paper, we introduce a simple and tractable framework to study these questions. Our theory highlights a tradeoff between commitment and risk management in borrowing long and short. Long-term debt has a staggered structure: it is not due soon and the borrower cannot commit to not issuing more debt before the existing one is due. Due to this lack of commitment, creditors of long-term debt are exposed to the risks of being diluted. By contrast, short-term debt does not suffer from dilution, because it matures very soon altogether and needs to be rolled over on a continuing basis. In other words, existing short-term debt must be retired before new one is issued. On the other hand, long-term debt has an important benefit of risk management: if a downturn arrives, losses in the enterprise value are shared between the borrower and long-term creditors, but not short-term creditors. This risk-sharing property of long-term debt increases the enterprise value, benefiting the borrower.

Let us be more specific. A risk-neutral borrower has assets in place, which generate an income flow that follows a geometric Brownian motion (GBM) process. The expected growth rate of the income is high in an upturn but low in a downturn. A transition from the upturn to the downturn can be interpreted as the downside risk. Creditors are competitive, risk-neutral, and have a lower cost of capital compared to the borrower. The difference in the cost of capital offers a reason for the borrower to issue debt, but too much debt could trigger default, in which case the asset has no recovery value. Two types of debt are available: short-term debt matures instantaneously

[1] Notable exceptions include He and Milbradt (2016) and DeMarzo and He (2021), which we discuss in the subsection on related literature.

(i.e., has zero maturity) and needs to be simultaneously rolled over, and long-term debt matures exponentially with a constant amortization rate. The key innovation of our model is to allow the borrower to have full flexibility in issuing either type of debt at any time to adjust the maturity profile of the outstanding debt. This feature differs us from the existing literature.

The flexibility to issue more debt exposes long-term creditors to dilution, due to the leverage-ratchet effect (Admati et al., 2018). Specifically, the borrower always has incentives to borrow more after existing long-term debt has been issued, because the additional borrowing dilutes the existing long-term claims. Note this is the case even though the asset has no value in default, because the additional debt pushes the borrower closer to default and reduces the value of long-term debt. In equilibrium, creditors anticipate the future dilution and the price of long-term debt adjusts downwards to the level that the borrower cannot capture any benefit. In other words, the borrower does not benefit from borrowing long even though creditors have a lower cost of capital.

By contrast, the instantaneous and simultaneous feature of short-term debt protects it from being diluted and resolves the commitment problem. Given that all the short-term debt needs to be rolled over on a continuing basis, existing short-term debt must be retired before any new debt is issued. In other words, for short-term creditors, their debt matures before the borrower can issue again and therefore does not suffer from dilution. As a result, short-term creditors need to be compensated only by the probability of default within a very short period of time, but not by the cost of being diluted in the future.

The advantage of short-term debt in resolving the commitment problem offers the borrower a natural reason to issue it. Indeed, our results show that long-term debt is never issued in the downturn, when there is no additional downside risk. Instead, the borrower fully levers up by borrowing short, which is riskless and does not suffer from dilution. Similar results hold in the upturn if the borrower is close to default. There, creditors anticipate default if the downturn arrives, so that short-term debt is not riskless. However, long-term debt is exposed not only to the same downside risk but also the dilution and hence more expensive.

Results are different in the upturn if borrower is far from default. Instead, the borrower issues both long- and short-term debt. In this case, creditors do not anticipate an immediate default if the downturn arrives, but the enterprise value still gets reduced. The reduced enterprise value is shared only between the borrower and long-term creditors, and short-term creditors' value left is intact. The drop in long-term debt's price highlights an important role of long-term debt in risk management: it allows the borrower to effectively make state-contingent payments. The state-contingent payments act as a cushion to reduce the borrower's burden in the downturn, mitigate the borrower's incentives to default, and eventually increase the enterprise value. Compared to long-term debt, short-term debt is a harder claim: the borrower must make non-state-contingent payments; otherwise she has to default. Note that the merit of long-term debt in sharing the downside risk is valued by the borrower, even though she is risk-neutral. Intuitively, the cost in default introduces constraints in financing, which makes the borrower behave as if she is risk-averse.

We show the enterprise value gets higher if the borrower is prohibited from borrowing long. The reason is, long-term debt does not benefit the borrower due to the lack of commitment, but, instead, introduces more defaults. Meanwhile, if the borrower is prohibited from borrowing short, the commitment problem becomes more severe: the enterprise value gets lower, so is the price of long-term debt. In other words, the ability to borrow short also increases the value of long-term debt. This result implies that long and short-term debt can be complements instead of substitutes.

Our model implies firms further away from default, more levered, and endowed with fewer growth options tend to use short-term debt, consistent with the empirical findings in Barclay and Smith Jr (1995). Moreover, the debt maturity structure is pro-cyclical, consistent with findings in Xu (2018) and Chen et al. (2021). An interesting prediction is the borrower responds differently to shocks at different frequency. Following frequent and small positive shocks to operating cash flows, the borrower tends to issue more short-term debt. By contrast, debt maturity is prolonged following infrequent and large positive shocks to the enterprise value.

### Related literature

Our paper builds on the literature of dynamic corporate finance, pioneered by Leland (1994). Most of this literature either fixes book leverage (Leland, 1998) or allows for adjustment with some issuance costs (Goldstein et al., 2001; Dangl and Zechner, 2020; Benzoni et al., 2019). Two important exceptions are DeMarzo and He (2021) and Abel (2018). Whereas the former studies leverage dynamics when the borrower has full flexibility in issuing exponentially-maturing debt, the latter addresses the same problem when the borrower can only issue zero-maturity debt. In both papers, the borrower can only issue one type of debt so the tradeoff between borrowing long and short is not explicitly studied.[2] Our paper fills the gap by combining the approaches and insights from both papers. He and Milbradt (2016) also study the problem of dynamic debt maturity management, where the total leverage is fixed and the borrower can choose between two types of exponentially-maturing debt. Our paper differs in two aspects. First, we allow for full flexibility in adjusting leverage. Second, we model short-term debt as one that matures simultaneously. The different approaches in modeling short-term debt render the mechanisms of the two papers drastically different: whereas we emphasize the tradeoff between commitment and risk-sharing,

[2] Malenko and Tsoy (2020) model the role of firm reputation under one type of debt, where maturity is chosen and fixed at the firm's origination. The tradeoff highlighted involves the feature that debt principal is not tax-deductible whereas interest payments are deductible. The channel is clearly different from us.

their paper focuses on rollover losses and dilution. Brunnermeier and Yogo (2009) also study debt maturity in the context of liquidity risk, and they show long-term debt is optimal if the firm is close to default (or debt restructuring in their paper). Our results are the opposite: the borrower will only borrow short if she is close to default. The difference is driven by the assumption that in our model, the borrower can issue debt at any time without commitment. By contrast, the borrower in Brunnermeier and Yogo (2009) can only issue new debt after the current one is repaid and effectively has commitment.

More broadly, our paper is related to the literature in corporate finance on debt maturity, starting from Flannery (1986) and Diamond (1991). This literature emphasizes the role of asymmetric information and the signaling role of short-term debt. The insight that short-term debt resolves lack of commitment is also present in another closely-related literature (Calomiris and Kahn, 1991; Diamond and Rajan, 2001) that emphasizes the runable feature of short-term debt. In our paper, the reason that why short-term debt resolves commitment is fundamentally different: the short rate would increase drastically had the borrower issued more debt.[3] We show short-term debt has the shortcoming of limited risk sharing. Relatedly, Gertner and Scharfstein (1991) show that conditional on financial distress, short-term debt has a higher market value and increases leverage, leading to more ex post debt overhang (also see Diamond and He (2014)).

The insight that long-term debt allows for more state-contingent payments when markets are incomplete is also present in the literature on sovereign debt. For example, Angeletos (2002) shows how the Arrow-Debreu allocation can be implemented with noncontingent debt of different maturities. According to Aguiar et al. (2019), the borrower never actively issues any long-term debt due to the lack of commitment. By contrast, we show the motives for risk sharing lead the borrowers to issue a combination of long and short. Arellano and Ramanarayanan (2012) consider maturity choice in a quantitative model of sovereign default with tradeoffs similar to ours. Our paper has two important differences. First, we fully characterize the optimal policy in debt maturity management. This characterization allows us to study the nature of the shocks that the borrower wants to hedge using long-term debt. Second, the borrower in our model is risk-neutral and therefore does not have a priori reason to value the merit of risk sharing by long-term debt. The cost of default makes the borrower to behave as if she is risk-averse, as emphasized by the corporate finance literature on risk management (Froot et al., 1993; Rampini and Viswanathan, 2010; Panageas, 2010). To the best of our knowledge, there is no previous work that establishes the link between maturity management and risk management.

[3] Also see Hu and Varas (2021) on this feature of short-term debt in the context of repo and shadow banking.

## 2 The Model

### 2.1 Agents and the Asset

Time is continuous and goes to infinity: $t \in [0, \infty)$. We study a borrower that can be either a firm or a government. In both cases, the relevant parties include the borrower as an equity holder and competitive creditors. Throughout the paper, we assume all agents are risk neutral, deep-pocketed, and protected by limited liability. Moreover, the borrower discounts the future at a rate $\rho$, which exceeds $r$, the discount rate of creditors.

During a short time interval $[t, t+dt)$, the borrower's asset generates income $X_t dt$, where

$$\frac{dX_t}{X_t} = \mu_{\theta_t} dt + \sigma dB_t, \tag{1}$$

$B_t$ is a standard Brownian motion, and $\theta_t \in \{H, L\}$ follows a two-state Markov chain, independent of $B_t$, with transition intensity $\lambda_{LH}$ and $\lambda_{HL}$, respectively. The drift $\mu_{\theta_t}$ differs across the two states with $\mu_L < \mu_H$, so that the high state $H$ is associated with a higher growth rate in the borrower's expected income.

### 2.2 Debt Maturity Structure

The difference between the discount rates $\rho - r$ offers benefits for the borrower to issue debt.[4] Throughout the paper, we allow the borrower to issue two types of debt, short and long, to adjust the debt maturity structure. In particular, we do not restrict the borrower to *commit* to a particular issuance path, but, instead, let the issuance decisions to be made at each instant.

All *short-term debt* matures *instantaneously* and *simultaneously* and needs to be rolled over continuously. We model short-term debt as one with zero maturity. Let $D_{t-} = \lim_{dt \downarrow 0} D_{t-dt}$ be the amount of short-term debt outstanding (and due) at time $t$ and $y_{t-}$ be the associated short rate. By contrast, *long-term debt* matures in a staggered manner. We follow the literature and model long-term debt as exponentially maturing coupon bonds with coupon rate $r$ and a constant amortization rate $\xi > 0$. Therefore, $1/\xi$ can be interpreted as the expected maturity. Let $F_t$ be the aggregate facevalue of long-term debt outstanding at time $t$ and $p_t$ the price per unit of the facevalue.

The borrower may default, in which case the bankruptcy is triggered. To isolate issues related to debt seniority and direct dilution in bankruptcy, we assume that the bankruptcy cost is 100%.

[4]The difference can be related to differences in liquidity, contracting costs or market segmentation. An alternative setup is to introduce tax shields, and the results are similar. In both setup, we take the debt contract as given and acknowledge that it can be the optimal solution under certain agency frictions.

In other words, creditors cannot recover any value once the borrower defaults.

### 2.3 Debt Price and Rollover

Let $T$ be the endogenous time at which the borrower defaults. For $t < T$, the price of the long-term debt per unit of facevalue satisfies

$$p_t = \mathbb{E}_t \left[ \int_t^{\tau_\xi \wedge T} e^{-r(s-t)} r ds + e^{-r(T-t)} \mathbb{1}_{\{T > \tau_\xi\}} \right], \tag{2}$$

where the two components in the expression correspond to the coupon and final payments. The short rate $y_t$ depends on the borrower's equilibrium default decisions:

$$y_t = r + \lim_{dt \downarrow 0} \frac{\Pr(T \leq t + dt | T \geq t)}{dt}, \tag{3}$$

where the second term on the right-hand side is the hazard rate of default. Clearly, $y_t = r$ whenever the short-term debt is default free. On the other hand, if default is predicted to happen deterministically at time $t + dt$, creditors will refuse to rollover short-term debt at $t$.[5] In general, $y_t$ compensates the creditors for the probability of default occurring between $t$ and $t + dt$. For example, if the borrower is anticipated to default following a transition from $H$ to $L$ state, $y_t = r + \lambda_{HL}$.

Over a short time interval $[t, t + dt)$, the net income to the borrower is

$$\left[ X_t - (r + \xi) F_t - y_{t-} D_{t-} \right] dt + p_t dG_t + dD_t, \tag{4}$$

where $(r + \xi) F_t$ is the interest and principal payments to long-term creditors, $y_{t-} D_{t-}$ the interest payments to short-term creditors. The remaining two terms $p_t dG_t$ and $dD_t$ are the proceeds from issuing long- and short-term debt.[6] Note that the notations $dG_t$ and $dD_t$ allow for both atomistic and flow issuance, and the price of long-term debt $p_t$ could also depend on the issuance amount $dG_t$.

Define $V_t$ as the continuation value of the borrower, which we sometimes refer to as the equity value at time $t$. The borrower chooses the endogenous time of default as well as the issuance of two types of debt to maximize the equity value, taking the price of long-term debt and the short-rate function as given. Once again, let us emphasize that all these decisions, default, and issuance, are

[5] Equivalently, $y_t \to \infty$. The default time $t$ is an accessible stopping time.

[6] One can think of $dD_t$ as the net issuance of short-term debt.

made without commitment.

$$V_t = \sup_{T,\{G_s,D_s:s\geq t\}} \mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)}\bigg(\left(X_s-(r+\xi)F_s-y_{s-}D_{s-}\right)ds+p_sdG_s+dD_s\bigg)\right]. \tag{5}$$

### 2.4 Smooth Equilibrium

The heuristic timing within a short time horizon $[t,t+dt)$ goes as follows.

1. The borrowers arrives at time $t$ with outstanding debt $\{D_{t-},F_t\}$.
2. The exogenous state $\theta_t$ is realized, and the borrower decides whether to repay or default on the outstanding debt.
   - If she defaults, the game ends, and nobody receives anything.
   - If she does not default, she repays $y_{t-}D_{t-}$ and $(r+\xi)\,F_t$ to short- and long-term creditors.
3. In the case of no default, the borrower receives income $X_t dt$. Moreover, she issues $dG_t$ long-term debt and $dD_t$ the net amount of short-term debt.[7]

We focus on the Markov perfect equilibrium (MPE) in which the payoff-relevant state variables include the exogenous state $\theta_t$, the income level $X_t$, and the amount of outstanding debt $\{D_{t-},F_t\}$. The equilibrium requires: 1) creditors break even, i.e., $p_t$ follows (2), and $y_t$ follows (3); 2) the borrower chooses optimal default and issuance (i.e., equation (5)), subject to the limited liability constraint $V_t\geq 0$. Finally, a MPE is *smooth* if there is no jump in long-term debt issuance, in which case we write $dG_t=g_tF_tdt$. In a smooth equilibrium, the aggregate facevalue of long-term debt evolves according to

$$dF_t=(g_t-\xi)\,F_tdt. \tag{6}$$

Let us define $J_t=V_t+D_{t-}$ as the joint continuation value of the borrower and creditors if default doesn't happen at time $t$. The following result motivates us to work with $J_t$ for the remainder of this paper.

[7] Specifically, $dD_t=D_t-D_{t-}$ if there is a jump at $t$.

**Lemma 1.** *The equity value equals* $V_t = V_{\theta_t}(X_t, F_t, D_{t-}) = \max\{J_t - D_{t-}, 0\}$, *where* $J_t = J_{\theta_t}(X_t, F_t)$ *is given by*

$$J_\theta(X,F) = \sup_{T,\{G_s, D_s : D_s \le J_\theta(X_s,F_s)\}} \mathbb{E}\left[\int_t^T e^{-(\rho+\lambda_{\theta\theta'})(s-t)}\Big(X_s - (r+\xi)F_s + (\rho + \lambda_{\theta\theta'} - y_{s-})D_{s-} \right.$$

$$\left. + \lambda_{\theta\theta'} \max\{J_{\theta'}(X_s, F_s) - D_{s-}, 0\}\Big)ds + p_s dG_s\Big) \middle| X_t = X,\ F_t = F,\ \theta_t = \theta\right]. \quad (7)$$

Lemma 1 suppresses the problem's dependence on $D_{t-}$. As in Abel (2018), as short-term debt can be freely adjusted, we can take it as a choice variable rather than a state variable. The characterization of the equilibrium based on the value function $J_\theta(X,F)$ is also related to the equilibrium characterization in Aguiar et al. (2019), who show that the equilibrium in a model of sovereign borrowing can be characterized by the solution of a planner's problem that ignores the value of the claims held by existing long-term debt holders. Therefore, we can take $\{X, F, \theta\} = \{X_t, F_t, \theta_t\}$ as the main state variables. A smooth Markov perfect equilibrium is characterized by functions $J_\theta(X,F)$, $p_\theta(X,F)$, $y_\theta(X,F)$, $D_\theta(X,F)$, and $g_\theta(X,F)$, where

$$J_\theta(X,F) = XJ_\theta\left(1, \frac{F}{X}\right) = Xj_\theta(f), \quad D_\theta(X,F) = XD_\theta\left(1, \frac{F}{X}\right) = Xd_\theta(f),$$

are homogeneous of degree one, and the rest are homogeneous of degree zero. Let $f = \frac{F}{X}$ be the long-term debt to income ratio, which is the *endogenous* state variable of the model. It follows from Itô's lemma that $f_t$ evolves according to

$$\frac{df_t}{f_t} = \left(g_{\theta_t}(f_t) - \xi - \mu_{\theta_t} + \sigma^2\right)dt - \sigma dB_t. \quad (8)$$

### 2.5 Modeling Discussion

**Risk and binary state.** The borrower faces two sources of risks. The Brownian motion captures fluctuations in day-to-day operating cash flows, which is meant to capture high-frequency risks. Whereas the expected growth rate in cash flow differs across the two exogenous states, transitions across the states have a more persistent impact. For the rest of the paper, we label the transition from the high state $H$ to the low state $L$ as *a downside risk*, which can be interpreted as lower-frequency, aggregate shocks occurring at either the industry- or the macroeconomy-level. The choice of binary state is made for tractability. Given that we focus on perspective of downside risk sharing, results in the high state are more relevant and should be interpreted more broadly.

**Debt maturity.** Our modeling choice of short- and long-term debt is motivated by the discrete-time microfoundation. There, short-term debt would last for one period and therefore mature simultaneously. In the continuous-time setup, this feature is captured by zero-maturity debt that needs to be continuously rolled over. In the discrete-time setup, long-term debt would last for multiple periods, and the flexibility in issuing them each period would lead to the staggered structure. This feature is well captured by exponentially-maturing debt in the continuous-time setup.

**Zero recovery in bankruptcy.** The assumption that creditors do not recover any value in bankruptcy is made without any loss of generality. As shown in DeMarzo (2019) and DeMarzo and He (2021), the borrower can effectively steal the liquidation value from creditors by issuing new debt just prior to default and using the proceeds to pay a dividend.[8] Moreover, this assumption rules out the theoretical channel highlighted in Brunnermeier and Oehmke (2013), where the equity holder directly dilutes existing creditors' recovery value in bankruptcy through issuing new debt.

**Parametric assumptions.** To make the problem non-trivial, we impose the following parametric restrictions. The first assumption enables us to obtain a tractable solution. Yet, as shown in section 4, the equilibrium doesn't rely on it.

**Assumption 1.**

$$\lambda_{LH} = 0, \qquad \lambda_{HL} = \lambda.$$

Assumption 1 says once the exogenous state is low, it will never return to the high state. By contrast, the state switches from high to low with a Poisson intensity $\lambda$. Thus negative shocks to the expected growth rate are permanent.

**Assumption 2.**

$$r + \lambda > \mu_H, \qquad r > \mu_L.$$

Assumption 2 is a standard one in the literature to guarantee that the valuation remains finite. Specifically, it requires in both states, the creditor's effective discount rate is above the expected growth rate of the cash flow.

**Assumption 3.**

$$\left(1 + \frac{\lambda}{\rho - r}\right)(\rho + \lambda - \mu_H) > (\rho + \lambda - \mu_L).$$

[8] Of course, this policy violates the absolute priority. See Dangl and Zechner (2020) for an analysis that takes into account of covenants that prevents such a dilution.

This final assumption introduces differences across the equilibrium in the two exogenous states. As we show below in subsection 3.1, it allows a material tradeoff when the borrower chooses between risky and riskless short-term debt in the high state. If this assumption is violated, the equilibrium results in the two states only differ quantitatively, and there will be fewer cases to analyze.

# 3 Equilibrium

Subsection 3.1 derives the value function and shows it can be solved as if the borrower never issues any long-term debt. We study how the borrower issues short-term debt in both states. Subsection 3.2 focuses on the issuance policy of long-term debt and explains the tradeoff behind borrowing long v.s. short. In subsection 3.3, we compare the equilibrium with one where only long- or short-term debt is allowed. Results there highlight the different roles of two types of debt. Finally, subsection 3.5 studies how an unlevered borrower issues long-term debt.

## 3.1 Value Function and Short-term Debt Issuance

**Low state $\theta_t = L$.** Under Assumption 1, the exogenous state $\theta_t$ will no longer change once it enters the low state $L$. Therefore, the remaining risk comes exclusively from the Brownian shock, and default can be anticipated by short-term creditors. As a result, short-term debt is riskless and demands a short rate $y_L(X,F) \equiv r$. By considering the change in the value function in (7) over a small interval, we can derive the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\underbrace{\rho J_L(X,F)}_{\text{required return}} = \max_{D_L \in [0, J_L(X,F)],\, g_L} \underbrace{X - (r+\xi)F}_{\text{cash flow net long payments}} + \underbrace{(\rho - r)D_L}_{\text{gains from borrowing short}} + \underbrace{p_L(X,F)\,g_L F}_{\text{proceeds from issuing long}} + \underbrace{\frac{\partial J_L(X,F)}{\partial F}(g_L - \xi)F}_{\text{evolution of } dF} + \underbrace{\frac{\partial J_L(X,F)}{\partial X}X\mu_L + \frac{1}{2}\frac{\partial^2 J_L(X,F)}{\partial X^2}X^2\sigma^2}_{\text{evolution of } dX}. \tag{9}$$

The net benefits of issuing long-term debt becomes clear once we examine all the terms that involve $g_L$ on the right-hand side. Whereas $p_L(X,F)$ captures the marginal proceeds from issuing an additional unit of long-term debt, $\frac{\partial J_L(X,F)}{\partial F}$ is the drop in the borrower's continuation value. If the borrower finds it optimal to adjust long-term debt smoothly, then it must be that the marginal proceeds are fully offset by the drop in continuation value, so that the borrower is indifferent, i.e.,

$$p_L(X,F) + \frac{\partial J_L(X,F)}{\partial F} = 0. \tag{10}$$

Under (10), the value function $J_L(X,F)$ can be solved as if $g_L \equiv 0$. We defer the characterization of long-term debt issuance until the next subsection. For now, let us plug (10) into (9) and use the fact that

$$\frac{\partial J_L(X,F)}{\partial F} = j_L'(f), \quad \frac{\partial J_L(X,F)}{\partial X} = j_L(f) - f j_L'(f), \quad X\frac{\partial^2 J_L(X,F)}{\partial X^2} = f^2 j_L''(f),$$

to get the following HJB for the scaled value function $j_L(f)$:

$$(\rho - \mu_L)\, j_L(f) = \max_{d_L \in [0, j_L(f)]} 1 - (r+\xi) f + (\rho - r)\, d_L - (\xi + \mu_L) f j_L'(f) + \frac{1}{2} f^2 j_L''(f)\,\sigma^2. \quad (11)$$

Turning to the issuance of short-term debt, whose net benefits are captured by the term $(\rho - r)\, d_L$ (or $(\rho - r)\, D_L$ in (9)). Intuitively, the creditor is more patient than the borrower, so that financing the enterprise by borrowing $d_L$ in short brings a net flow benefit of $(\rho - r)\, d_L$. Given $\rho > r$, it is always optimal for the borrower to lever up using as much short-term debt as possible, which leads to $d_L = j_L(f)$ (or equivalently $D_L = J_L(X,F)$) under the limited liability constraint.

The rest of the problem is standard. We look for a solution to (11) on $f \in \left[0, f_L^b\right]$, where the endogenous default boundary $f_L^b$ satisfies the value-matching condition $j_L\left(f_L^b\right) = 0$ and smooth-pasting condition $j_L'\left(f_L^b\right) = 0$. Proposition 1 describes the equilibrium outcome.

**Proposition 1** (Equilibrium when $\theta_t = L$)**.** *In the unique equilibrium, the value function is*

$$j_L(f) = \underbrace{\frac{1}{r - \mu_L} - f}_{\text{no default value}} + \underbrace{\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^{\gamma}}_{\text{default option value}}, \quad (12)$$

*where $\gamma > 1$. The default boundary is $f_L^b = \frac{\gamma}{\gamma-1}\frac{1}{r-\mu_L}$. For $\forall f \in [0, f_L^b)$, the borrower issues short-term debt $d_L(f) = j_L(f)$ and pays a short rate $y_L(f) = r$.*

**High state $\theta_t = H$.** The smooth equilibrium leads to an indifference condition in long-term debt issuance that relates to (10)

$$p_H(X,F) + \frac{\partial J_H(X,F)}{\partial F} = 0.$$

Besides the Brownian shock, there is the additional downside risk, whereby the state may transit from high to low. If default does not occur right upon the transition, the borrower and short-term creditor receive a maximum value of $j_L(f)$, among which $d_H$ is repaid to short-term creditors. Clearly, the borrower will default right upon the state transition if and only if $d_H > j_L(f)$. Expecting so, short-term creditors demand a short rate

$$y_H(f, d_H) = \begin{cases} r & \text{if } d_H \le j_L(f) \\ r + \lambda & \text{if } d_H > j_L(f). \end{cases} \tag{13}$$

Following a similar analysis to the one in the low state, we arrive at the HJB for the scaled value function $j_H(f)$:

$$\begin{aligned} (\rho + \lambda - \mu_H) j_H(f) = \max_{0 \le d_H \le j_H(f)} & 1 - (r + \xi) f + (\rho - r) d_H + \mathbb{1}_{\{d_H \le j_L(f)\}} \cdot \lambda j_L(f) \\ & - (\xi + \mu_H) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f). \end{aligned} \tag{14}$$

Compared to (11), there are two differences. First, the borrower's effective discount rate becomes $\rho + \lambda$, due to the possibility of state transition. Second, the term $\mathbb{1}_{\{d_H \le j_L(f)\}} \cdot \lambda j_L(f)$ captures the scenario that with intensity $\lambda$, the downside risk is realized, upon which the borrower and short-term creditors receive a continuation payoff $j_L(f)$ if and only if $d_H \le j_L(f)$. Otherwise, the borrower defaults and both receive nothing. Note that the flow benefit of short-term debt is still captured by $(\rho - r) d_H$ and in particular does not include the credit risk premium $\lambda$, even in the case that short-term debt is risky (i.e., $d_H > j_L(f)$). Intuitively, the risk premium serves as a transfer between the borrower and short-term creditors and therefore does not enter the joint continuation value.

The optimal issuance of short-term debt is straightforward: the borrower borrows either $j_L(f)$ at rate $r$ or $j_H(f)$ at $r + \lambda$. The choice between the two entails a price-quantity tradeoff: whereas $d_H = j_L(f)$ is cheap for the borrower, $d_H = j_H(f)$ allows the borrower to borrow more at a higher rate. Equation (14) can be therefore written as

$$\begin{aligned} (\rho + \lambda - \mu_H) j_H(f) = 1 - (r + \xi) f - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) \\ + \max\Big\{(\rho - r) j_L(f) + \lambda j_L(f), (\rho - r) j_H(f)\Big\}, \end{aligned} \tag{15}$$

where the term $\max\Big\{(\rho - r) j_L(f) + \lambda j_L(f), (\rho - r) j_H(f)\Big\}$ again captures the tradeoff in borrowing short. If the equity holder borrows riskless short-term debt, the flow benefit is lower $[(\rho - r) j_L(f) < (\rho - r) j_H(f)]$. However, because there is no immediate default after the state transition, the borrower avoids the expected bankruptcy cost $\lambda j_L(f)$. Assumption 3 ensures $(\rho - r) j_L(0) + \lambda j_L(0) > (\rho - r) j_H(0)$, so that it is optimal to borrow riskless short-term debt when $f = 0$. On the other hand, when $f$ gets close to $f_L^b$, $j_L(f)$ is very low, and the borrower can only borrow a small amount of riskless short-term debt. In this case, she would rather borrow the risky but higher amount $j_H(f)$. We show in Lemma 2 of the appendix that there exists a unique threshold $f_\dagger \in \left(0, f_L^b\right)$ such that $(\rho + \lambda - r)\, j_L(f) \le (\rho - r)\, j_H(f)$ if and only if $f \ge f_\dagger$. Given so, the HJB becomes a second-order ordinary differential equation (ODE) on both $[0, f_\dagger]$ and $\left[f_\dagger, f_H^b\right]$. The solutions and the two free boundaries $\left\{f_\dagger, f_H^b\right\}$ are pinned down by six boundary conditions: 1) value-matching and smooth-pasting at $\left\{f_\dagger, f_H^b\right\}$; 2) Transversality condition at $f = 0$; and 3) the indifference between issuing risky and riskless short-term debt at $f_\dagger$. The detailed expressions are available in (26) to (31) in the appendix. Proposition 2 describes the equilibrium in $H$, where the constant $\phi$ and the expressions for $\left\{u_0(f), u_1(f), h_0\left(f, f_\dagger, f_H^b\right), h_1\left(f, f_\dagger, f_H^b\right)\right\}$ are in the appendix.

**Proposition 2** (Equilibrium when $\theta_t = H$)**.** *In the unique equilibrium, the value function is*

$$j_H(f) = \begin{cases} u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^\phi & f \in [0, f_\dagger] \\ u_1(f) + \left(j_H(f_\dagger) - u_1(f_\dagger)\right) h_0\left(f, f_\dagger, f_H^b\right) + u_1(f_H^b) h_1\left(f, f_\dagger, f_H^b\right) & f \in \left[f_\dagger, f_H^b\right], \end{cases} \tag{16}$$

*where $\left\{f_\dagger, f_H^b\right\}$ solve equations (32) and (33) in the appendix. For $\forall f \in [0, f_H^b)$, the borrower issues short-term debt*

$$d_H(f) = \begin{cases} j_L(f) & \text{if } f \le f_\dagger \\ j_H(f) & \text{if } f > f_\dagger \end{cases}$$

*and pays a short rate given by equation (13).*

### 3.2 The Issuance of Long-term Debt

The previous subsection shows that the borrower is indifferent between issuing long-term debt or not in the smooth equilibrium. However, the result doesn't necessarily imply that she won't borrow long on the equilibrium path. In this subsection, we solve for the issuance policy of long-term debt, starting from state $\theta_t = L$.

**Low state $\theta_t = L$.** Equation (10) (or equivalently $p_L(f) = -j_L'(f)$) is the necessary condition for the borrower to be indifferent between issuing long-term debt or not. Meanwhile, the price satisfies the following HJB equation

$$(r + \xi)\, p_L(f) = \underbrace{r + \xi}_{\text{coupon and principal}} + \underbrace{\left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)}_{\text{expected change in bond price}}. \tag{17}$$

To derive the issuance function $g_L$, we plug $d_L = j_L(f)$ into (11), differentiate the resulting equation once, and add (17).

**Proposition 3** (Long-term debt issuance when $\theta_t = L$)**.** *For $\forall f \in [0, f_L^b)$, $g_L(f) = 0$, and the long-term debt's price is $p_L(f) = -j_L'(f)$.*

**High state $\theta_t = H$.** Similarly, the equity holder's indifference in long-term debt issuance implies $p_H(f) = -j_H'(f)$, and $p_H(f)$ satisfies the following HJB equation

$$(r + \xi + \lambda) p_H(f) = r + \xi + \mathbb{1}_{\{f \le f_\dagger\}} \lambda p_L(f) + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f). \tag{18}$$

Compared to (17), (18) includes the additional event of state transition, upon which the price drops to $p_L(f)$ if $f \le f_\dagger$; otherwise, the borrower defaults and the price drops to zero. The derivation of $g_H$ follows similar steps.

**Proposition 4** (Long-term debt issuance when $\theta_t = H$)**.** *For $\forall f \in [0, f_H^b)$, the rate of long term issuance is given by*

$$g_H(f) = \begin{cases} \frac{(\rho - r)\left(p_H(f) - p_L(f)\right)}{-f p_H'(f)} & f \le f_\dagger \\ 0 & f > f_\dagger. \end{cases} \tag{19}$$

*and the long-term debt's price is $p_H(f) = -j_H'(f)$.*

Proposition 3 shows that in the low state, the equity holder never issues any long-term debt, but, instead, borrows the maximum amount of short-term debt. By contrast, Proposition 4 shows long-term debt is issued in the high state when the amount of short-term borrowing is riskless, i.e., $f \le f_\dagger$. Why might the equity holder borrow long in the high state but not in low? Why would the equity holder borrow long in the high state only if the amount of short-term borrowing is riskless? What are the differential roles of short- and long-term debt?

Due to the leverage-ratchet effect, the equity holder faces a time-inconsistency problem when borrowing long: she is unable to commit to a path of issuance, but, instead, always has incentives to issue more and dilute legacy long-term creditors. As shown by (10), this lack of commitment implies the borrower is unable to capture any benefits from borrowing long, even though the creditors are more patient. On the other hand, short-term debt, in particular its combined nature of instantaneous maturity and simultaneity, resolves the commitment problem, because all outstanding debt must be rolled over continuously, i.e., the existing short-term debt must be retired before issuing any new one. Given that short-term debt is riskless and cheap ($y_L = r$) in the low state, one would imagine that the equity holder only borrows short.[9] Similar results hold in the high state when short-term debt is risky, i.e., when $f > f_\dagger$. Given that the borrower is expected to default upon the state transition, short-term creditors demand a short rate $r + \lambda$. Meanwhile, long-term debt is subject not only to the same downside risk of state transition but also the leverage ratchet/dilution effect. Therefore, the equity holder, again, only borrows short.

Matters are different in the high state when short-term debt is riskless, i.e., when $f \leq f_\dagger$. In this case, default does not occur after the downside risk is realized, but the enterprise value experiences a discontinuous jump then. Whereas a transition from the high to the low state reduces the equity value by $j_H(f) - j_L(f)$ and the long-term debt price by $p_H(f) - p_L(f)$, it leaves the value of short-term debt intact. In other words, the loss in enterprise value is shared between the borrower and long-term creditors, short-term creditors, on the other hand, do not share any loss. This result highlights an important role of long-term debt in risk sharing: long-term debt allows the borrower to make state-contingent payment ($p_H(f)$ in $H$ but $p_L(f)$ in $L$) without default. The state-contingent payments act as a cushion which reduces the borrower's burden in the low state and mitigates the incentives to default, thereby increasing the enterprise value. By contrast, short-term debt is a *harder* claim: the borrower must make non-state-contingent payments; otherwise she has to default. A careful examination of the issuance function (19) shows that the amount of long term debt issued, $fg_H$, equals the ratio of the benefits from risk-sharing $(\rho - r)\big(p_H(f) - p_L(f)\big)$ to the price sensitivity of new issuance $-p_H'(f)$.[10]

Thus, the choice of maturity is determined by the trade-off between commitment and risk sharing. Short-term debt resolves the issue of lack of commitment; long-term debt shares the downside risk. This insight relates to the previous work in sovereign debt literature that emphasizes how long-term debt allows for more state-contingency (Angeletos, 2002). A key difference is, we cast the model in the context of a risk-neutral borrower, as it is typically the case in capital structure studies.

Why would a risk-neutral borrower value the long-term debt's merit in sharing the downside risk? The reason is, the bankruptcy cost introduces concavity into her objective function, so that she behaves as if risk-averse. As emphasized by previous work on risk management (Froot et al., 1993; Rampini and Viswanathan, 2010), a risk-neutral corporation has incentives to insure or hedge

[9]Note that the result of no long-term debt issuance in the low state stays unchanged even if $\lambda_{LH} > 0$, because the borrower will not default upon a state transition from low to high.

[10]Interestingly, the borrower may actually buy back/repurchase long-term debt before it matures. Examining equation (19), we know long-term debt repurchase occurs whenever $p_H(f) < p_L(f)$, which might be the case when $f$ is only slightly below $f_\dagger$. Intuitively, long-term debt in this region is riskier in the high state, because a combination of a bad cash-flow shock $dB_t$ and the downside risk will lead to the borrower default, whereas in the low state, the borrower is still reasonably far from the default boundary.

against negative shocks when the cost of external financing is costly and fluctuates. In our context, a borrower with outstanding long-term debt $\{X_t, F_t, D_{t-}\} = \{X, F, D\}$ seeks to maximize

$$\underbrace{V\left(X, F', D\right)}_{\text{equity value}} + \underbrace{\left(D' - D\right) + p\left(X, F'\right)\left(F' - F\right)}_{\text{issuance proceeds}}$$

by choosing $\{D', F'\}$. Let $f' = \frac{F'}{X}$ and $d' = \frac{D'}{X}$, the borrower equivalently maximize

$$j\left(f'\right) + p\left(f'\right)\left(f' - f\right), \tag{20}$$

which is the sum of equity holder and new creditors' payoff $j\left(f'\right) + p\left(f'\right) f'$. By contrast, existing long-term creditor's claims do not enter this objective function at all. Figure 1 plots this objective function for $f = \frac{f_\dagger}{2}$, and clearly, the function is concave in $f'$.

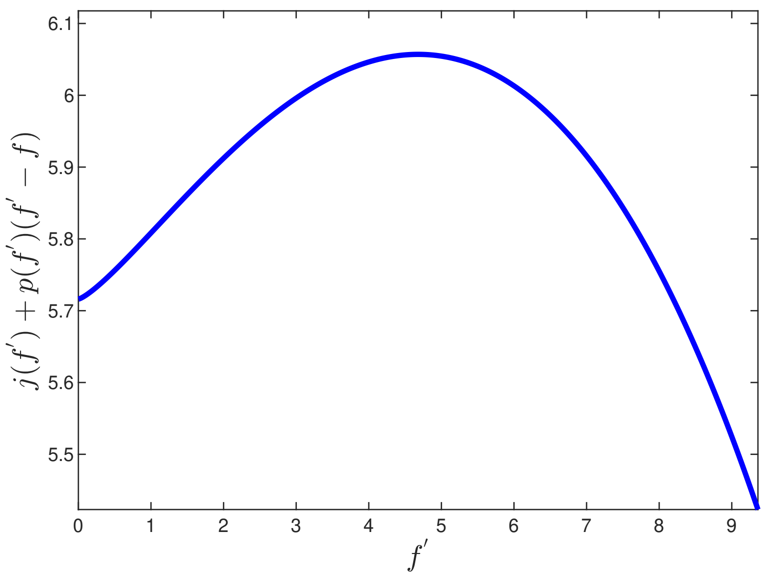


**Figure 1: Borrower's Objective Function** (20)

This figure plots the objective function (20) as a function of $f'$, for $f = \frac{f_\dagger}{2} = 4.68$. The other parameters are as follows: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$.

We can illustrate the previous trade-off by looking at the marginal rate of substitution between short- and long-term debt. Suppose the borrower's outstanding long-term debt is $f \leq f_\dagger$. If the borrower increases its long-term borrowing by a small amount $\Delta f$, while keeping short-term debt as riskless, then she needs to reduce short-term borrowing by $j_L(f) - j_L(f + \Delta f)$. Such a policy generates an additional cash flow of $p_H(f+\Delta)\Delta + j_L(f+\Delta) - j_L(f)$. For $\Delta f$ sufficiently small, this extra cash flow is approximately $[p_H(f) - p_L(f)]\Delta f$, which is positive if $p_H(f) > p_L(f)$. Therefore, for $f \le f_\dagger$, the additional proceeds from issuing a small amount of long-term debt exceeds the drop in proceeds from issuing risk-less short-term debt, thereby benefiting the borrower. On the other hand, suppose the borrower's outstanding long-term debt is $f > f_\dagger$. The same increase in long-term borrowing will reduces the amount of short-term debt by $j_H(f) - j_H(f+\Delta f)$,[11] changing cash flows by $p_H(f+\Delta f)\Delta f + j_H(f+\Delta f) - j_H(f)$. The impact on cash flow is negative for any $\Delta f > 0$ and is approximately zero as $\Delta f$ converges to zero.[12] Therefore, for $f > f_\dagger$, any additional proceeds from issuing long-term debt is more than offset by the drop in proceeds from issuing short-term debt.

Our result shows a pecking order among the choice of the riskiness and maturity of debt. In particular, short-term riskless debt is the most favored, as illustrated by the borrower's choice in the low state. If short-term debt may not be riskless, as the case in the high state, the borrower starts to borrow long-term debt that is subject to the potential of borrower default and tries to maintain the riskless feature of short-term debt. This is the situation in the high state when $f < f_\dagger$. Furthermore, if the amount of outstanding long-term debt gets too high such that short-term debt can never be riskless, the borrower turns to borrowing exclusively short-term risky debt and in the mean time reduces the outstanding long-term debt by not issuing it.

### 3.3 Benefits and Costs of Long- and Short-term Debt

In this subsection, we explore the benefits and costs of both types of debt by studying the equilibrium if only long or short-term debt is allowed.

**Proposition 5** (Equilibrium with only short-term debt)**.** *There is an unique equilibrium if the borrower can only issue short-term debt.*

1. *In the low state L, the borrower never defaults. The value function is* $\tilde{J}_L(X) = \frac{X}{r-\mu_L}$ *and the borrower issues short-term debt* $\tilde{D}_L = \tilde{J}_L(X)$ *at rate* $y_L = r$.
2. *In the high state H,*

[11] Note that the short-term debt cannot be risk-less for $f > f_\dagger$.

[12] The proof goes as follows.

$$p_H(f+\Delta f)\Delta f + j_H(f+\Delta f) - j_H(f) = p_H(f+\Delta f)\Delta f + j_H'\left(f+\tilde{\Delta} f\right)\Delta f$$
$$\left[p_H(f+\Delta f) - p_H\left(f+\tilde{\Delta} f\right)\right]\Delta f < 0,$$

where $\tilde{\Delta} f \in (0, \Delta f)$. The first equality follows from Mean Value Theorem. The rest follows from the necessary condition of smooth issuance and the fact that $p_H$ decreases with $f$.

(a) *If* $\frac{1}{r+2\lambda-\mu_H} > \frac{1}{\rho+\lambda-\mu_H} + \frac{\rho-r+\lambda}{\rho+\lambda-\mu_H}\frac{1}{r-\mu_L}$, *the borrower defaults upon the state transition. The value function is* $\tilde{J}_H(X) = \frac{X}{r+2\lambda-\mu_H}$, *and the borrower issues short-term debt* $\tilde{D}_H = \tilde{J}_H(X)$ *at rate* $y_H = r+\lambda$.

(b) *Otherwise, the borrower never defaults. The value function is* $\tilde{J}_H(X) = \frac{X}{\rho+\lambda-\mu_H} + \frac{\rho-r+\lambda}{\rho+\lambda-\mu_H}\frac{X}{r-\mu_L}$, *and the borrower issues short-term debt* $\tilde{D}_H = \tilde{J}_L(X)$ *at rate* $y_H = r$.

3. *The total enterprise value is higher than the case if the borrower can issue both types of debt, i.e.,* $\tilde{J}_\theta(X) \geq J_\theta(X,F) + p_\theta(X,F)F$, $\forall\theta \in \{L,H\}$.

If only short-term debt is allowed, the commitment problem in debt issuance no longer exists. Instead, the choice of capital structure is a static problem and follows the standard trade-off theory whereby equity holders balance cheap debt against costly bankruptcy. Given that the problem is scalable with respect to $X_t$, the solution is one with a constant leverage level. Interestingly, the total enterprise value is higher if the borrower is prohibited from issuing long-term debt.[13] Intuitively, long-term debt does not benefit the borrower at all due to the lack of commitment, but, instead, introduces the possibility of default and the associated bankruptcy cost if the cash flow gets sufficiently low (or equivalently $f$ gets very high).

**Proposition 6** (Equilibrium with only long-term debt)**.** *There is a unique equilibrium.*

1. *In state* $L$, *the value function is*

$$\tilde{v}_L(f) = \frac{1}{\rho-\mu_L} - \frac{r+\xi}{\rho+\xi}f + \frac{r+\xi}{\rho+\xi}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}},$$

*where the default boundary is* $\tilde{f}_L^b = \frac{1}{\rho-\mu_L}\frac{\tilde{\gamma}}{\tilde{\gamma}-1}\frac{\rho+\xi}{r+\xi}$.

2. *In state* $H$, *the value function is*

$$\tilde{v}_H(f) = \tilde{u}_0(f) - \tilde{u}_0\left(\tilde{f}_H^b\right)\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi}.$$

*The borrower defaults upon the state transition if and only if* $f > \tilde{f}_L^b$.

3. *The total enterprise value is lower than the case if the borrower can issue both types of debt, i.e.,* $\tilde{v}_\theta(f) + \tilde{p}_\theta(f)f \leq j_\theta(f) + p_\theta(f)f$, $\forall f$.

[13] In fact, if the borrower has any outstanding long-term debt, the enterprise value is strictly lower than the case without any outstanding long-term debt!

4. *In both states $\theta \in \{L, H\}$, the debt price is $\tilde{p}_\theta = -\tilde{v}'_\theta$, and the issuance function follows $\tilde{g}_\theta = \frac{(\rho - r)\tilde{p}_\theta}{-f\tilde{p}'_\theta}$. The debt prices satisfy $\tilde{p}_L < p_L$, $\forall f \in [0, \tilde{f}^b_L]$ and $\tilde{p}_H < p_H$ for $f$ either sufficiently low or sufficiently high.*

When the borrower is only allowed to issue long-term debt, the setup resembles the one in DeMarzo and He (2021). Without commitment to the issuance policy, equity holders do not reap the benefits from issuing cheaper debt, as the debt price will adjust for the future issuance policy. In equilibrium, long-term debt is issued smoothly. In both states, the enterprise value is higher when the borrower can issue both types of debt. Intuitively, borrowing short not only increases the leverage but also allows the equity holder to reap some benefits issuing debt.

The price of long-term debt is in general higher when the borrower can issue both types of debt. The reason is, whereas short-term debt increases the enterprise value, it also pushes up the default boundary, so that under the same level of long-term debt, the borrower is further away from the default boundary. This result suggests that long-term debt and short-term debt could complement each other, as opposed to being substitutes.[14]

### 3.4 Downside Risk

In the benchmark model, the borrower is subject to two types of risks. The Brownian motion captures small frequent shocks to the cash flow, which has a continuous effect on the enterprise value continuously. By contrast, a transition from the high to the low state, i.e., the regime shift, captures large infrequent shocks that reduce the enterprise value discontinuously. In this subsection, we show the modeling choice of regime shift is unimportant. In particular, our mechanism continues to hold if large infrequent shocks are modeled as downward jump risks to the cash flow. Specifically, let us assume the cash flow follows a jump-diffusion process:

$$dX_t = \mu X_{t-} dt + \sigma X_{t-} dB_t - \left(1 - \eta^{-1}\right) X_{t-} dN_t, \tag{21}$$

where $N_t$ is a Poisson process with intensity $\lambda$ and $\eta \in (1, \infty)$ is a constant. We can construct an equilibrium characterized by thresholds $f_\dagger$ and $f^b$. The issuance of long-term debt satisfies $g(f) = 0$, for $g(f) = 0\ \forall f \in (f_\dagger, f^b]$, where $f^b$ is the endogenous default boundary. For $f \in [0, f_\dagger]$,

[14] In the high state when $f$ is close to $f_\dagger$, the price of long-term debt can be lower if she can issue both debt. Intuitively, taking short-term debt leads to the borrower default following the realization of the downside risk when $f$ is above $f_\dagger$. Without short-term debt, the borrower won't default following the same transition unless $f$ rises above $\tilde{f}^b_L$. Therefore, the price of long-term debt is relatively low when $f$ exceeds $f_\dagger$ but is still far from $\tilde{f}^b_L$ yet.

the issuance of long-term debt follows

$$g(f) = \frac{(\rho - r)(p(f) - p(\eta f))}{f j''(f)}. \tag{22}$$

In other words, long-term debt is issued if and only the amount of outstanding long-term debt is low relative to the operating cash flow. Examining (22), it becomes clear that the intuitive reason again fall into the benefits of long-term debt in sharing downside risks, modeled as downward jumps in this case. The difference in prices $p(f) - p(\eta f)$ reflects the drop in the long-term debt's price following the downward jump, and the expression thus can be similarly interpreted as (19).

The issuance of short-term debt is also similar to that in section 3. Short-term debt is riskless when $f \leq f_\dagger$ and the amount of issuance is $d(f) = j(\eta f)$. On the other hand, when $f > f_\dagger$, short-term debt becomes risky, and the amount of issuance becomes $d(f) = j(f)$. The scaled-value function $j(f)$ satisfies a second order delay differential equation, which cannot be solved in closed form. Detailed analysis is available in appendix A.3.

### 3.5 Initial Debt Issuance

By applying L'Hôpital's Rule to the issuance function (19), we can show

$$g_H(0) = \frac{(\rho - r)(\mu_H - \mu_L)}{\rho + \lambda - r}, \qquad \lim_{f \to 0} g_H(f) f = 0,$$

which implies an unlevered borrower does not issue any long-term debt. The intuition is straightforward. For the unlevered borrower, both $p_H(f) \to 1$ and $p_L(f) \to 1$ hold as $f \to 0$, so that a marginal unit of long-term debt is riskless and does not share any downside risk. Therefore, the borrower does not want to issue it. Indeed, the issuance function (19) makes it clear that, for an unlevered borrower to issue long-term debt, it must be $p_H(0) > p_L(0)$ so that a marginal unit of long-term debt shares some losses following the transition from the high to the low state.

There are several approaches to motivate an unlevered borrower to issue long-term debt. One is to introduce an exogenous disaster in the low state, modeled as a Poisson event with intensity $\zeta$, upon which $X_t$ can permanently drop to zero. In this case, both $p_H(0)$ and $p_L(0)$ are less than 1 (and therefore not default free) and $p_H(0) > p_L(0)$.

**Proposition 7.** *In the model with the disaster event, $g_L = 0$ and $g_H(f)$ is described by (19). If the constants $\phi_1$ and $\gamma_1$ defined in the appendix satisfy $\min\{\phi_1, \gamma_1\} \geq 2$ and $\zeta < \frac{\lambda}{\rho - r}(\rho + \lambda - \mu_H) - (\mu_H - \mu_L)$, then $\lim_{f \to 0} g_H(f) f > 0$.*

Why would an unlevered borrower issue long-term debt in the high state when she faces a disaster risk in the low state? The reason, again, goes back to the role of long-term debt in sharing downside risk. If the downside risk only includes the state transition from high to low, the marginal unit of long-term debt is riskless for an unlevered borrower. Now that there is the additional downside risk from the disaster, a marginal unit of long-term debt is more exposed to the disaster risk in the low state compared to itself in the high state, even if the borrower is unlevered. Mathematically, $p_H(f) - p_L(f) > 0$ holds even as $f \to 0$. This exercise confirms that an unlevered borrower has incentives to issue long-term debt as long as the marginal unit shares some downside risk.

# 4 Empirical Implications

In this section, we explore the model's empirical implications. For this purpose, we numerically solve the version of the model with the disaster event introduced in subsection 3.5. Moreover, we relax Assumption 1 and instead allow for $\lambda_{LH} > 0$. Therefore, shocks to the cash flow's expected growth rate are transitory. Under empirically reasonable parameters, we solve for an equilibrium similar to the one in section 3. We are going to derive cross-sectional and time-series implications on debt maturity structure and link them to empirical studies.

Our central object of interest is a firm's debt maturity structure, defined as the average maturity of total debt outstanding, weighted by the book value:

$$\text{Maturity}_t := \frac{F_t}{F_t + D_t}\frac{1}{\xi} = \frac{f_t}{f_t + d_t}\frac{1}{\xi}. \tag{23}$$

Figure 2 plots how the average maturity varies within a cross-section of borrowers with different characteristics. The left panel plots how the maturity changes with $f$ in the high state under different levels of $\mu_L$.[15] One interpretation of $f$ is the borrower's distance to default (DD), and a higher $f$ means the borrower being closer to default. Two patterns are prominent. First, a borrower closer to default has more long-term debt. In our model, this pattern holds because in the absence of regime shift, default is only triggered by a large amount of outstanding long-term debt. A borrower closer to default can only issue a low amount of short-term debt. Second, for a given $f$, the borrower whose cash flows grow at a lower rate in the low state (captured by a lower $\mu_L$) uses more long-term debt. Intuitively, this borrower's incentives to hedge against

[15] We argue the results in the high state are a more precise description of a firm, for the following reason. In the binary state setup, there is no additional downside risk once the borrower enters the low state. In practice, it is likely that the borrower always faces some downside risk, which motivates a reason to use long-term debt. In other words, the high state in our model is meant to capture any real-world scenario as long as the borrower still faces some downside risk.

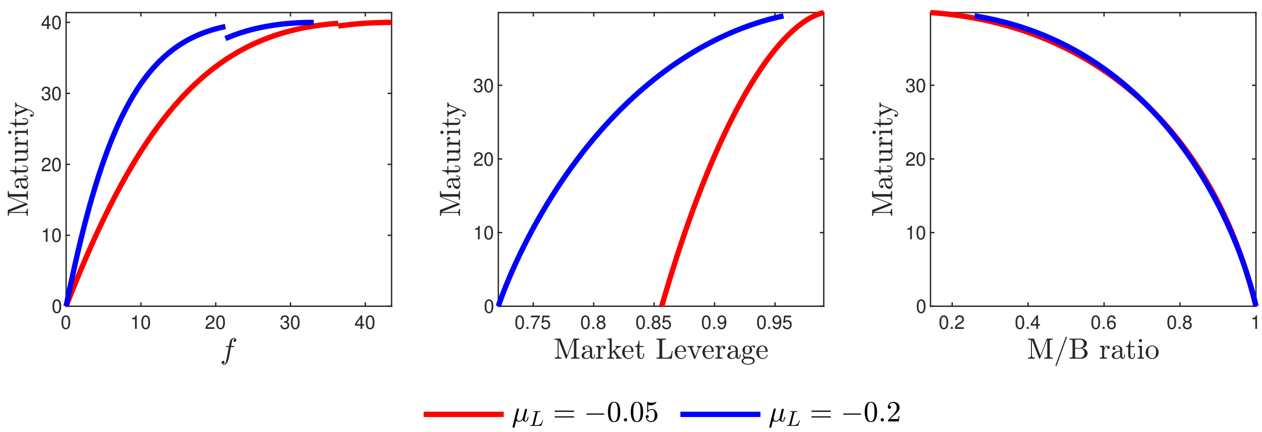


**Figure 2: Cross-sectional Debt Maturity Structure**

This figure plots maturity as a function of $f$, market leverage and market-to-book ratio in the high state. The parameters are as follows: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\sigma = 0.3$, $\xi = 0.025$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$. The first figure plots maturity as a function of $f$ on $[0, f_H^b]$. The second and third figure plot maturity as a function of leverage and market to book ratio for $f$ on $[0, f_\dagger]$.

the downside risk are more prominent.[16] In the middle panel, we show how maturity differs across firms with different leverage, where leverage is defined as the book value of total debt divided by the sum of market value of equity and book value of debt $\frac{d+f}{j+f}$.[17] This measurement corresponds to the market leverage ratio in most empirical papers.[18] Results show that more levered borrowers use more long-term debt. Moreover, a comparison between the two lines shows that the borrower with higher downside risk uses more long-term debt as well. Finally, the right panel plots maturity across firms with different asset market to book ratio, defined as $\frac{p(f)f+j}{f+j(f)}$. Clearly, a firm with more growth options uses more short-term debt. All these empirical results are consistent with prior empirical findings, such as Stohs and Mauer (1996) and Barclay and Smith Jr (1995).

To characterize the time series implications, we analyze the stochastic process of the average

[16] Similar patterns hold for higher $\lambda_{HL}$, due to the same intuition reason.

[17] The leverage is 100% in the low state and in the high state when $f > f_\dagger$, implying that firms with the same level of high leverage (100% in this case) could have different maturity structures.

[18] For example, empirical papers using Compustat data tend to define market leverage of firm $i$ in year $t$ as

$$\text{Lev}_{it} = \frac{\text{DLTT}_{it} + \text{DLC}_{it}}{\text{DLTT}_{it} + \text{DLC}_{it} + \text{CSHO}_{it} \times \text{PRCC\_F}_{it}},$$

where $\text{DLTT}_{it}$ and $\text{DLC}_{it}$ are the amount of long-term debt and debt in current liabilities. PRCC_F is the fiscal year-end common share price and CSHO is the fiscal year-end number of shares outstanding.

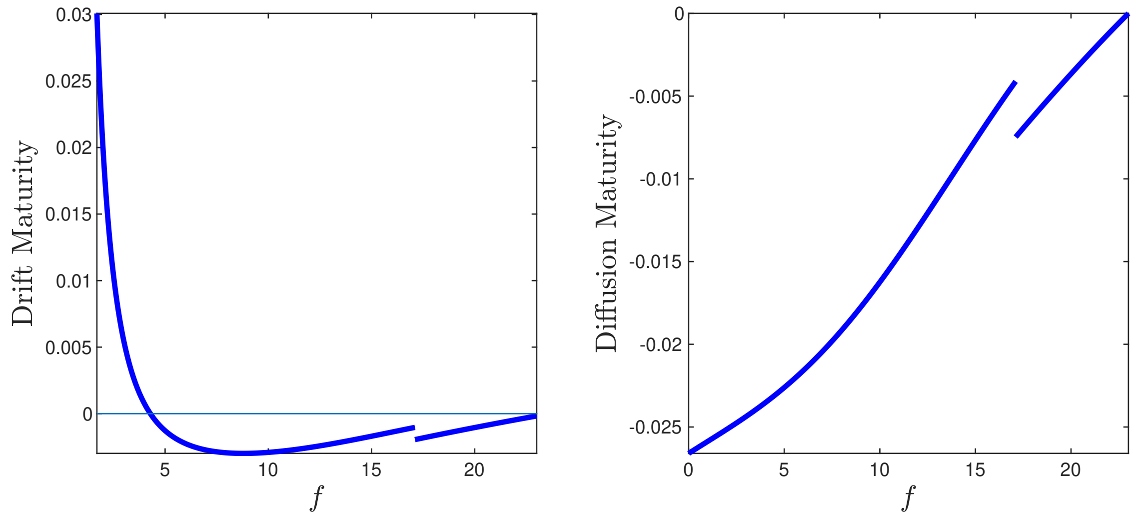


**Figure 3: Drift and Diffusion of Maturity Structure**

This figure plots the drift and diffusion of maturity in the high state when $f < f_\dagger$ with the following parameter values: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$.

maturity by applying Itô's Lemma to (23). In the low state, the borrower never issues any long-term debt and seeks to fully finance the enterprise using short-term debt. Therefore, the maturity structure moves towards all short-term debt as the low state persists. This result is consistent with the ample empirical evidence that suggests downgraded firms mainly borrow short-term debt. Whereas the literature has mostly explained this falling-angel phenomenon from the perspective of financiers, our theory offers an alternative story from the perspective of the borrower.[19] The left panel of Figure 3 plots the drift in the high state, which captures the relative issuance of long- and short-term debt. Comparing the drifts in both states, our model implies that the debt maturity structure is, on average, procyclical, consistent with the findings in Chen et al. (2021). Moreover, long-term debt is mostly issued in the good state, consistent with Xu (2018), who shows that speculative-grade are actively lengthening their debt maturity structure in good times via early refinancing. Finally, the drift crosses the horizontal axis once, implying that a borrower has a target maturity structure.

The right panel of Figure 3 shows the diffusion coefficient in the high state, which captures the impulse response of average maturity to small and frequent shocks to the borrower's incoming cash flows. The diffusion coefficient is always negative, so that a positive cash-flow shock encourages the firm to issue relatively more short-term debt. This shocks results in a negative correlation between

[19] Of course this is an equilibrium argument: for the equity holder, borrowing long is too expensive in the low state.

the maturity structure and operating cash flows. Intuitively, a positive shock to the cash flow $X_t$ increases the enterprise value and allows the borrower to immediately issue more short-term debt, whereas long-term debt can only be adjusted smoothly. Indeed, the flexibility in borrowing short allows the borrower to quickly capture the benefits from small and frequent cash flow shocks. On the other hand, the borrower ceases to borrow long-term debt following the regime shift, which is a large and infrequent shock to the total enterprise value. Taken together, our results show that a firm's maturity structure responds differently to different shocks. More short-term debt is issued following small and frequent positive shocks to the operating cash flows. However, more short-term debt is also issued after infrequent but large negative shocks, such as the downside risk which presumably occurs at the business-cycle frequency.

We simulate a sample path and plots the time-series of debt maturity and market leverage in Figure 4. The top panel plots the sequences of cash flow rate $X_t$ and maturity. Again, one can see that the borrower responds by prolonging the debt maturity structure following negative shocks to $X_t$, and in downturns, debt maturity structure is in general lower compared to upturns. The bottom panel shows that market leverage is on average counter-cyclical, if we interpret the state transition as business-cycle shocks. Adrian and Shin (2014) offer consistent evidence.

# 5 Conclusion

Our paper offers a theory of debt maturity, which is fundamentally a tradeoff between commitment and risk sharing. Short-term debt mitigates the lack of commitment problem but does not share any downside risk. Long-term debt suffers from dilution but shares the downside risk. In a model with binary state, risk sharing is not valued in the downturn or in the upturn if the borrower is close to default. If the borrower is far from default in the upturn, she borrows both debt.

We haven't modeled callable bonds and exchange offers, which are common tools for corporate firms to manage debt maturity in practice. In our model, if long-term debt can be frictionlessly called back, then there is no essential difference between long- and short-term debt, and debt maturity becomes irrelevance. The assumption of frictionless callback is not an innocuous simplification, and further exploration towards this direction goes beyond the scope of this paper. In practice, another motive behind debt maturity management is to match assets and liabilities, which we intend to study in follow-up work.

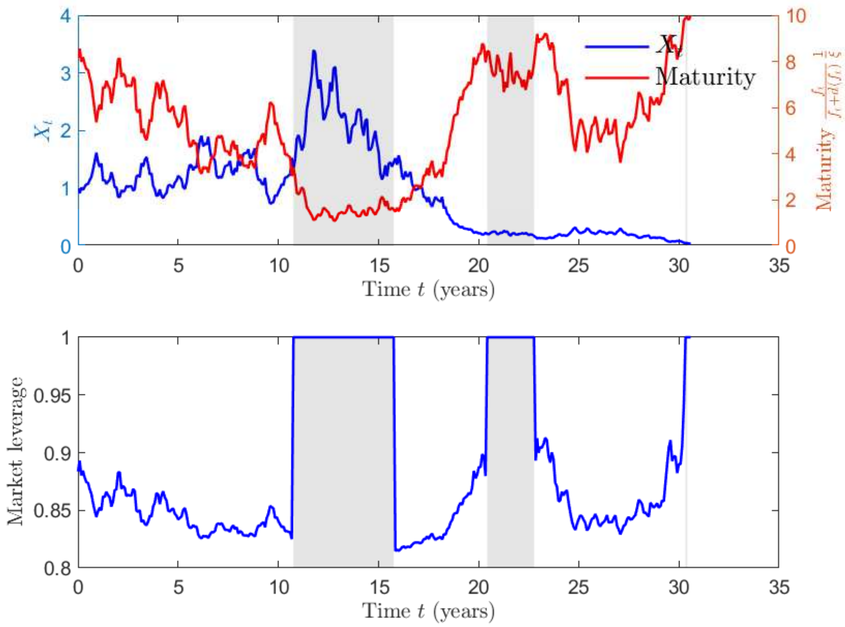


**Figure 4: Sample Path of Leverage and Maturity**

This figure simulates the sample path of one firm and plots the time series of $X_t$, maturity, and market leverage, with the following parameter values: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$.

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# A Appendix

### Proof of Lemma 1

*Proof.* Let $\tau \geq t$ be the time that the state switches from $\theta$ to $\theta'$. By the principle of dynamic programming,

$$\begin{aligned}
V_t &= \sup_{T,\{G_s,D_s:s\in[t,\tau)\}} \mathbb{E}_t\Bigg[\int_t^{T\wedge\tau} e^{-\rho(s-t)}\Big( (X_s-(r+\xi)F_s-y_{s-}D_{s-})\,ds + p_s dG_s + dD_s\Big) + e^{-\rho T\wedge\tau}V_\tau \mathbb{1}_{\{T\geq\tau\}}\Bigg] \\
&= \sup_{T,\{G_s,D_s:s\in[t,\tau)\}} \mathbb{E}_t\Bigg[\int_t^{T} e^{-(\rho+\lambda)(s-t)}\Big( (X_s-(r+\xi)F_s-y_{s-}D_{s-})\,ds + p_s dG_s + dD_s\Big) + e^{-\rho\tau}V_\tau \mathbb{1}_{\{T\geq\tau\}}\Bigg].
\end{aligned}$$

Given the definition of $J_t$, the equity value can be written as $V_t = \max\{J_t - D_{t-}, 0\}$, where the max operator takes into account the borrower's limited liability constraint. In particular, the borrower defaults at time $\tau$ if $J_\tau < D_{\tau-}$, so that $V_\tau = \max\{J_\tau - D_{\tau-}, 0\}$. Hence,

$$\begin{aligned}
V_t = &\sup_{T,\{G_s,D_s:s\in[t,\tau)\}} \mathbb{E}_t\Bigg[\int_t^{T} e^{-(\rho+\lambda)(s-t)}\Big( (X_s-(r+\xi)F_s-y_{s-}D_{s-})\,ds + p_s dG_s + dD_s\Big) \\
&\qquad + e^{-\rho\tau}\max\{J_\tau - D_{\tau-},0\}\,\mathbb{1}_{\{T\geq\tau\}}\Bigg] \\
= &\sup_{T,\{G_s,D_s:s\in[t,\tau)\}} \mathbb{E}_t\Bigg[\int_t^{T} e^{-(\rho+\lambda)(s-t)}\Big( \big(X_s-(r+\xi)F_s-y_{s-}D_{s-}+\lambda_{\theta_s\theta_s'}\max\{J_s-D_{s-},0\}\big)\,ds \\
&\qquad + p_s dG_s + dD_s\Big)\Bigg]
\end{aligned}$$

Using the integration by parts formula for semi-martingales in Corollary 2 in Section 2.6 of Protter (2005), we get

$$\mathbb{E}_t\Bigg[\int_t^T e^{-(\rho+\lambda)(s-t)}dD_s\Bigg] = \mathbb{E}_t\Big[e^{-(\rho+\lambda)(T-t)}D_T\Big] - D_{t-} + \mathbb{E}_t\Bigg[\int_t^T e^{-(\rho+\lambda)(s-t)}(\rho+\lambda)D_{s-}ds\Bigg].$$

At the time of default, $D_T = 0$. Hence

$$\begin{aligned}
V_t = &\sup_{T,\{G_s,D_s:s\in[t,\tau)\}} \mathbb{E}_t\Bigg[\int_t^{T} e^{-(\rho+\lambda)(s-t)}\Big( \big(X_s-(r+\xi)F_s+(\rho+\lambda-y_{s-})D_{s-}+\lambda_{\theta_s\theta_s'}(J_s-D_{s-})^+\big)\,ds \\
&\qquad + p_s dG_s + dD_s\Big)\Bigg] - D_{t-}.
\end{aligned}$$

□

## A.1 Maximum Principle

Our proofs use repeatedly the Maximum Principle for differential equations. Theorem 3 and 4 from Chapter 1 in Protter and Weinberger (1967) are particularly useful, and we state them below.

**Theorem 1** (Theorem 3 in Protter and Weinberger (1967))**.** *If $u(x)$ satisfies the differential inequality*

$$u'' + g(x)u' + h(x)u \geq 0 \tag{24}$$

*in an interval $(0, b)$ with $h(x) \leq 0$, if $g$ and $h$ are bounded on every closed subinterval, and if $u$ assumes a nonnegative maximum value $M$ at an interior point $c$, then $u(x) \equiv M$.*

**Theorem 2** (Theorem 4 in Protter and Weinberger (1967))**.** *Suppose that $u$ is a nonconstant solution of the differential inequality (24) having one-sided derivatives at $a$ and $b$, that $h(x) \leq 0$, and that $g$ and $h$ are bounded on every closed subinterval of $(a, b)$. If $u$ has a nonnegative maximum at $a$ and if the function $g(x) + (x - a)h(x)$ is bounded from below at $x = a$, then $u'(a) > 0$. If $u$ has a nonnegative maximum at $b$ and if $g(x) - (b - x)h(x)$ is bounded from above at $x = b$, then $u'(b) > 0$.*

**Corollary 1.** *If $u$ satisfies (24) in an interval $(a, b)$ with $h(x) \leq 0$, if $u$ is continuous on $[a, b]$, and if $u\,(a) \leq 0$, $u\,(b) \leq 0$, then $u\,(x) < 0$ in $(a, b)$ unless $u \equiv 0$.*

## A.2 Proof of Section 3

### Proof of Proposition 1

*Proof.* Equation (11) is a second-order ODE, and a standard solution takes the form

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Plugging into the ODE, we can get

$$\begin{aligned}
(r - \mu_L)\, A_0 &= 1 \\
(r + \xi)\, A_1 &= (r + \xi) \\
(r - \mu_L)\, A_2 &= -\,(\mu_L + \xi)\, A_2 \gamma_1 + \frac{1}{2}\sigma^2 A_2 \gamma_1 \,(\gamma_1 - 1) \\
(r - \mu_L)\, A_3 &= -\,(\mu_L + \xi)\, A_3 \gamma_2 + \frac{1}{2}\sigma^2 A_3 \gamma_2 \,(\gamma_2 - 1)\,,
\end{aligned}$$

which implies

$$A_0 = \frac{1}{r - \mu_L},\ A_1 = 1$$

and $\{\gamma_1, \gamma_2\}$ are the two roots of

$$\frac{1}{2}\sigma^2 \gamma_1^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\gamma_1 - (r - \mu_L) = 0.$$

In particular,

$$\gamma_1 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r - \mu_L)}}{\sigma^2} > 1,$$
$$\gamma_2 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r - \mu_L)}}{\sigma^2} < 0.$$

The condition

$$\lim_{f \to 0} j_L(f) < \infty$$

implies $A_3 = 0$. Therefore, we define

$$\gamma \equiv \gamma_1 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r - \mu_L)}}{\sigma^2}.$$

Combining with value-matching and smooth-pasting condition, we get the solution to $j_L(f)$ and $f_L^b$.

To derive $g_L(f)$, let us first write down the HJB for $p_L(f)$:

$$(r + \xi)\, p_L(f) = (r + \xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f), \tag{25}$$

where we have used the condition (8). The result of $g_L(f) \equiv 0$ follows from differentiating (11), applying in condition (10), and subtracting (25). □

**Proof of Proposition 2**

*Proof.* Let us first write down the HJBs in different regions as well as the boundary conditions. Specifically, the value function satisfies

$$(\rho+\lambda-\mu_H)\,j_H(f) = 1-(r+\xi)\,f+(\rho+\lambda-r)\,j_L(f)-(\mu_H+\xi)\,fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f),\ \ f\in[0,f^\dagger]$$
$$(r+\lambda-\mu_H)\,j_H(f) = 1-(r+\xi)\,f-(\mu_H+\xi)\,fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f),\ \ f\in[f^\dagger,f_H^b].$$

The boundary conditions are

$$j_H(f\dagger-) = j_H(f\dagger+) \tag{26}$$
$$j_H'(f_H^{b-}) = j_H'(f_H^{b+}) \tag{27}$$
$$j_H(f_H^b) = 0 \tag{28}$$
$$j_H'(f_H^b) = 0 \tag{29}$$
$$\lim_{f\to 0}\ j_H(f) < \infty \tag{30}$$
$$j_H\left(f_\dagger\right) = \left(1+\frac{\lambda}{\rho-r}\right)j_L\left(f_\dagger\right). \tag{31}$$

Next, let us supplement the expressions of the auxiliary functions. The solutions to the constants $\{\phi,\beta_1,\beta_2\}$ and the boundaries $\left\{f_\dagger,f_H^b\right\}$ are provided as we solve the ODE system.

$$u_0(f) \equiv \frac{1}{r-\mu_L}\frac{\rho+\lambda-\mu_L}{\rho+\lambda-\mu_H}-f+\frac{(\rho+\lambda-r)}{(\rho+\lambda-r)+(\mu_H-\mu_L)\,(\gamma-1)}\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^\gamma$$
$$u_1(f) \equiv \frac{1}{r+\lambda-\mu_H}-\frac{r+\xi}{r+\xi+\lambda}f$$
$$h_0\left(f,f_\dagger,f_H^b\right) = \frac{\left(\frac{f}{f_H^b}\right)^{\beta_1}-\left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}$$
$$h_1\left(f,f_\dagger,f_H^b\right) = \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\left(\frac{f}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}-\frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}.$$

The rest of the proof includes thre parts. In the first part, we detail the solutions to the ODE system (15) combined with the boundary conditions (26)-(31). In the second part, we prove a single-crossing property and therefore shows that it is optimal for the borrower to issue riskless short-term debt $d_H = j_L(f)$ if and only if $f \le f_\dagger$. Finally, we verify that $j_H(f)$ is a convex function on $\left[0, f_H^b\right]$, so that it is indeed optimal for the borrower to issue long-term debt smoothly.

**Part 1: the solution to the ODE system.** On $[0, f_\dagger]$, the solution to the ODE taking into condition (30) shows that

$$j_H\left(f\right) = u_0(f) + Bf^{\phi},$$

where

$$\phi = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho + \lambda - \mu_H\right)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2\phi^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\phi - \left(\rho + \lambda - \mu_H\right) = 0.$$

The coefficient $B$ is pinned down from the value at $j_H\left(f_\dagger\right)$

$$B = f_\dagger^{-\phi}\left(j_H(f_\dagger) - u_0(f_\dagger)\right)$$

so that

$$j_H(f) = u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^{\phi}, \qquad \forall f \in [0, f_\dagger],$$

where $j_H(f_\dagger) = \left(1 + \frac{\lambda}{\rho - r}\right)j_L(f_\dagger)$.

On $\left[f_\dagger, f_H^b\right]$, the solution to the ODE is

$$j_H\left(f\right) = u_1(f) + D_1 f^{\beta_1} + D_2 f^{\beta_2},$$

where

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2} > 1$$

$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda\right)}}{\sigma^2} < 0$$

solve

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta - (r + \lambda - \mu_H) = 0.$$

Using (28) and (26), we get

$$D_1 = \frac{j_H(f_\dagger) + u_1(f_H^b)\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - u_1(f_\dagger)}{(f_H^b)^{\beta_1}\left[\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\right]}$$
$$D_2 = (f_H^b)^{-\beta_2}\left(-u_1(f_H^b) - D_1(f_H^b)^{\beta_1}\right).$$

so that

$$u_1(f) + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)h_0\left(f, f_\dagger, f_H^b\right) + u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right).$$

It remains to find equations that solve $\left\{f_\dagger, f_H^b\right\}$, which come from the smooth pasting conditions (27) and (29). These two conditions lead to the two-variable, non-linear equation system below

$$u_1(f_H^b)\left[\frac{\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right] = u_1'(f_H^b)f_H^b + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \tag{32}$$

$$\begin{aligned}\big(u_0'(f_\dagger) - u_1'(f_\dagger)\big)f_\dagger + \phi\big(j_H(f_\dagger) - u_0(f_\dagger)\big) &= \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \\ &\quad + u_1(f_H^b)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2}.\end{aligned} \tag{33}$$

**Part 2: single-crossing.** The following result shows that it is optimal for the borrower to issue $d_H = j_L(f)$ when $f \leq f_\dagger$ and $d_H = j_H(f)$ otherwise.

**Lemma 2** (Single-crossing)**.** *There exists a unique* $f_\dagger \in \left(0, f_L^b\right)$ *such that* $(\rho + \lambda - r)j_L(f) \geq (\rho - r)j_H(f)$ *if and only if* $f \leq f_\dagger$.

*Proof.* Define $a \equiv 1 + \frac{\lambda}{\rho - r}$. The goal is to show $aj_L - j_H > 0$ for $f < f_\dagger$, and vice versa. Let us introduce two operators: for a function $u$ let,

$$\begin{aligned}L^{0\dagger}u &\equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H+\xi)\, f u' - (\rho+\lambda-\mu_H)\, u\\ L^{\dagger b}u &\equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H+\xi)\, f u' - (r+\lambda-\mu_H)\, u.\end{aligned}$$

The HJB in state $\theta = H$ (15), can be therefore written as

$$\begin{aligned}L^{0\dagger} j_H + 1 - (r+\xi)\, f + (\rho+\lambda-r)\, j_L = 0,\ \ f \in (0, f^\dagger)\\ L^{\dagger b} j_H + 1 - (r+\xi)\, f = 0,\ \ f \in (f^\dagger, f_H^b).\end{aligned}$$

Similarly, the HJB in state $\theta = L$, (11) can be written as

$$\begin{aligned}L^{0\dagger} a j_L + a(\mu_H-\mu_L) f j_L' + a\,(\rho+\lambda-r+\mu_L-\mu_H)\, j_L + a\,(1-(r+\xi)\, f) = 0\\ L^{\dagger b} a j_L + a(\mu_H-\mu_L) f j_L' - a(\mu_H-\mu_L-\lambda) j_L + a\,(1-(r+\xi)\, f) = 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}L^{0\dagger}\,(a j_L - j_H) + H(f) = 0\\ L^{\dagger b}\,(a j_L - j_H) + H(f) = 0,\end{aligned}$$

where the function $H(f)$ is defined as

$$H(f) \equiv a(\mu_H-\mu_L) f j_L' - a(\mu_H-\mu_L-\lambda) j_L(f) + (a-1)\,(1-(r+\xi)\, f)\,,$$

and satisfies

$$\begin{aligned}H''(f) &= \Big[(\mu_H-\mu_L) a \frac{f j_L'''}{j_L''} + (\mu_H-\mu_L) a + (\rho+\lambda-r)\,(a-1)\Big] j_L''\\ &= \Big[(\mu_H-\mu_L) a(\gamma-1) + (\rho+\lambda-r)\,(a-1)\Big] j_L'' > 0.\end{aligned}$$

Therefore, $H\,(f)$ is a convex function, and its maximum on $[0, f_L^b]$ is attained on the boundary 0 or $f_L^b$. Evaluating $H(f)$ at the two boundaries and using Assumption 3, we get

$$\begin{aligned}H(0) &= \frac{a(\rho+\lambda-\mu_H) - (\rho+\lambda-\mu_L)}{r-\mu_L} > 0\\ H(f_L^b) &= (a-1)\left(1-(r+\xi)\, f_L^b\right) < 0.\end{aligned}$$

Therefore, there exists a unique $f'$ such that $H(f) \geq 0$ on $[0, f']$ and $H(f) \leq 0$ on $[f', f_L^b]$. Depending on whether $f' < f_\dagger$ or not, we need to consider two cases.

- Case 1: $f' > f_\dagger$.
  - On $f \in [0, f_\dagger]$, we know $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f_\dagger]$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f_\dagger)$. Moreover, Theorem 2 and Corollary 1 imply $aj'_L(f_\dagger) - j'_H(f_\dagger) < 0$.
  - On $f \in [f', f_L^b]$, we know $H(f) \leq 0$ and $L^{\dagger b}(aj_L - j_H) \geq 0$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Given that $aj_L\left(f_L^b\right) - j_H\left(f_L^b\right) < 0$, $aj_L(f) - j_H(f) \leq 0$, $\forall f \in \left[f', f_L^b\right]$.
  - On $f \in [f_\dagger, f']$. Suppose there exists a $f'' \in (f_\dagger, f')$ such that $aj_L(f'') - j_H(f'') > 0$. Given that $aj_L(f_\dagger) - j_H(f_\dagger) = 0$ and $aj'_L(f_\dagger) - j'_H(f_\dagger) < 0$, it must be that $aj_L(f) - j_H(f)$ has a nonpositive interior minimum on $[f_\dagger, f'']$. Meanwhile, from $L^{\dagger b}(aj_L(f) - j_H(f)) \leq 0$ for $f \in (f_\dagger, f'')$, we know from Theorem 1 that $aj_L(f) - j_H(f)$ cannot have a nonpositive interior minimum on $(f_\dagger, f'')$, which constitutes a contradiction.
- Case 2: $f' \leq f_\dagger$.
  - On $f \in [f_\dagger, f_L^b]$, we know that $H(f) < 0$ and $L^{\dagger b}(aj_L - j_H) \leq 0$. From Theorem 1 and 2, we know $aj_L(f) - j_H(f) \leq 0$ and $aj'_L(f_\dagger) - j'_H(f_\dagger) \leq 0$.
  - On $f \in [f', f_\dagger]$, $L^{0\dagger}(aj_L - j_H) \geq 0$ so that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Together with $aj'_L(f_\dagger) - j'_H(f_\dagger) \leq 0$, this shows $aj_L(f) - j_H(f) \geq 0$.
  - On $f \in [0, f']$, we know that $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f']$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f')$.

$\square$

**Part 3: the convexity of $j_H(f)$ on $\left[0, f_H^b\right]$.** The proof relies on a few auxiliary lemmas.

**Lemma 3.**

$$j'_H(f) \geq -1, \qquad \forall f \in \left[0, f_H^b\right],$$

*Proof.* Let $\hat{u} = j'_H(f) + 1$ and the goal is to show $\hat{u}(f) \geq 0$, $\forall f \in \left[0, f_H^b\right]$. We know from (16) that $\hat{u}(0) = 0$ and (29) that $\hat{u}\left(f_H^b\right) = 1$. Moreover, $\hat{u}$ satisfies

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (\rho + \lambda + \xi)\,\hat{u} = -(\rho + \lambda - r)\left(j'_L + 1\right) < 0, \qquad f \in [0, f_\dagger]$$
$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (r + \lambda + \xi)\,\hat{u} = -\lambda < 0 \qquad f \in [f_\dagger, f_H^b].$$

By Theorem 1, we know $\hat{u}(f)$ cannot admit a nonpositive interior minimum on $\left[0, f_H^b\right]$, which rules out the possibility that $\hat{u}(f) < 0$. □

**Lemma 4.**

$$f_H^b > \frac{1}{r + \xi} \qquad \textit{and} \qquad \min\left\{j''_H(0), j''_H\left(f_H^b\right)\right\} > 0,$$

*Proof.* For any $f \leq \frac{1}{r+\xi}$, there is a naive policy that the equity holder does not issue any long-term debt, in which case the scaled net cash flow rate becomes $1 - (r + \xi) f + (\rho + \lambda - y) d > 0$. In other words, the naive policy generates positive cash flow to the borrower, so that it is never optimal to default. Therefore, it must be that $f_H^b > \frac{1}{r+\xi}$. Plugging (28) and (29) into (15), we get $j''_H\left(f_H^b\right)$ whenever $f_H^b > \frac{1}{r+\xi}$.

Next, let us turn to prove that $j''_H(0) \geq 0$. Let us define $u \equiv j'_H$ and differentiate the HJB once

$$\frac{1}{2}\sigma^2 f^2 u'' - \left(\mu_H + \xi - \sigma^2\right) f u' - (\rho + \lambda + \xi)\,u = (r + \xi) - (\rho + \lambda - r)\,j'_L.$$

Moreover, let $z$ be the solution to

$$\frac{1}{2}\sigma^2 f^2 z'' - \left(\mu_H + \xi - \sigma^2\right) f z' - (\rho + \lambda + \xi)\,z = (r + \xi) - (\rho + \lambda - r)\,j'_L(0)$$

with boundary conditions

$$\lim_{f \downarrow 0} z(f) < \infty$$
$$z(f_\dagger) = u(f_\dagger) = j'_H(f_\dagger).$$

The solution is

$$z(f) = -\frac{r + \xi}{\rho + \lambda + \xi} + \frac{(\rho + \lambda - r)\,j'_L(0)}{\rho + \lambda + \xi} + \left(j'_H(f_\dagger) + \frac{r + \xi}{\rho + \lambda + \xi} - \frac{(\rho + \lambda - r)\,j'_L(0)}{\rho + \lambda + \xi}\right)\left(\frac{f^{\omega_1}}{f_\dagger}\right)^{\omega_1},$$

where

$$\omega_1 = \frac{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho + \lambda + \xi\right)}}{\sigma^2} > 0.$$

Let $\delta(f) = z - u$. It is easily verified that $\delta(0) = 0$ and $\delta(f_\dagger) = 0$. Moreover, $\delta$ satisfies

$$\frac{1}{2}\sigma^2 f^2 \delta'' - \left(\mu_H + \xi - \sigma^2\right) f\delta' - \left(\rho + \lambda + \xi\right)\delta = \left(\rho + \lambda - r\right)\left(j_L'(f) - j_L'(0)\right) \geq 0.$$

By Theorem 1, $\delta$ cannot have an interior nonnegative maximum, and the maximum is attained at $f = 0$. Theorem 2 further implies $\delta'(0) < 0$ so $u'(0) > z'(0)$. Finally, we know that

$$z'(f) = \omega_1\left(j_H'(f_\dagger) + \frac{r+\xi}{\rho+\lambda+\xi} - \frac{\left(\rho+\lambda-r\right)j_L'(0)}{\rho+\lambda+\xi}\right) f_\dagger^{-\omega_1} f^{\omega_1-1} = \omega_1\left(j_H'(f_\dagger) + 1\right) f_\dagger^{-\omega_1} f^{\omega_1-1},$$

which implies $z'(f) \geq 0$ given that $j_H'(f_\dagger) \geq -1$. Therefore, $u'(0) = j_H''(0) > 0$. □

**Lemma 5.**

$$j_H'''(f_\dagger^-) > j_H'''(f_\dagger^+).$$

*Proof.* We differentiate the HJB (15) once and take the difference between the left limit $f_\dagger-$ and right limit $f_\dagger+$

$$\frac{1}{2}\sigma^2 f^2 (j_H'''(f_\dagger+) - j_H''''(f_\dagger-)) = (\rho - r)\left[a j_L'(f_\dagger) - j_H'(f_\dagger)\right],$$

where $a \equiv 1 + \frac{\lambda}{\rho - r}$ The proof of Proposition 2 shows $a j_L'(f_\dagger) - j_H'(f_\dagger) < 0$ so that

$$j_H''''(f_\dagger-) > j_H'''(f_\dagger+).$$

□

To finish the rest of the proof, we differentiate the HJB twice and let $\tilde{u} \equiv f j_H''$

$$\left(\rho + \lambda + \xi\right)\tilde{u} = \left(\rho + \lambda - r\right) f j_L'' - \left(\mu_H + \xi - \sigma^2\right) f\tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in [0, f_\dagger] \tag{34}$$

$$\left(r + \lambda + \xi\right)\tilde{u} = -\left(\mu_H + \xi - \sigma^2\right) f\tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in \left[f_\dagger, f_H^b\right]. \tag{35}$$

By the maximum principle in Theorem 1, $\tilde{u}$ cannot have an interior nonpositive local minimum in $(0, f_\dagger) \cup (f_\dagger, f_H^b)$. Because $\tilde{u}$ is differentiable on $(0, f_\dagger) \cup (f_\dagger, f_H^b)$, the only remaining possibility of a nonpositive minimum is that $\tilde{u}(f_\dagger) < 0$. As $\tilde{u}(0)$ and $\tilde{u}(f_H^b)$ are positive, this requires that

$j''_H(f_\dagger-) + f_\dagger j'''_H(f_\dagger-) = \tilde{u}'(f_\dagger-) < \tilde{u}'(f_\dagger+) = j''_H(f_\dagger+) + f_\dagger j'''_H(f_\dagger+)$. From the HJB equation it follows that $j_H$ is twice continuously differentiable at $f_\dagger$, so such a kink would require $j'''_H(f_\dagger^-) < j'''_H(f_\dagger^+)$, which is ruled out by Lemma 5. We can conclude that $\tilde{u}$ does not have an interior nonpositive minimum, so it follows that $\tilde{u}(f) = f j''_H(f) > 0$ on $(0, f_H^b)$. $\square$

**Proof of Proposition 5**

*Proof.* In the low state, short-term debt is riskless and the borrower never defaults. The firm value is $\tilde{J}_L(X) = \frac{X}{r-\mu_L}$, and the borrower borrows 100% short-term debt.

In the high state, there is a choice between borrowing risky and riskless debt. If she borrows risky short-term debt, again, she would like to take 100% leverage. In this case, the firm value is $\tilde{J}_H(X) = \frac{X}{r+2\lambda-\mu_H}$. On the other hand, if she borrows riskless debt up to $X_t j_L$, the firm value becomes $\hat{J}_H(X) = \frac{X}{\rho+\lambda-\mu_H} + \frac{\rho-r+\lambda}{\rho+\lambda-\mu_H}$.

Finally, $\tilde{J}_L(X) \geq J_L(X,F) + p_L(X,F)F$ is straightforward given the former is the first-best firm value. In the high state, this is equivalent to proving $\tilde{j}_H \geq j_H(f) + p_H(f)f$. It is easily verified that $\tilde{j}_L \geq j_L(0)$ and the equality only holds in the second case where $\tilde{D}_H = \tilde{J}_L(X) < \tilde{J}_H(X)$, and the result follows from $\frac{d[j_H(f)+p_H(f)f]}{df} = p'_H(f)f < 0$. $\square$

**Proof of Proposition 6**

*Proof.* In the low state, the HJB becomes

$$\rho \tilde{V}_L = X - (r+\xi)F - \frac{\partial \tilde{V}_L}{\partial F}\xi F + \frac{\partial \tilde{V}_L}{\partial X}\mu_L X + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 \tilde{V}_L}{\partial X^2}.$$

Again, let $\tilde{V}_L = X\tilde{v}_L$ so that

$$\begin{aligned}\frac{\partial \tilde{V}_L}{\partial F} &= \tilde{v}'_L \\ \frac{\partial \tilde{V}_L}{\partial X} &= \tilde{v}_L - f\tilde{v}'_L \\ X\frac{\partial^2 \tilde{V}_L}{\partial X^2} &= f^2\tilde{v}''_L.\end{aligned}$$

The scaled HJB becomes

$$(\rho-\mu_L)\tilde{v}_L = 1 - (r+\xi)f - (\mu_L+\xi)f\tilde{v}'_L + \frac{1}{2}\sigma^2 f^2 \tilde{v}''_L.$$

Using the conditions

$$\lim_{f\to 0} \tilde{v}_L(f) < \infty$$
$$\tilde{v}_L\left(\tilde{f}_L^b\right) = 0$$
$$\tilde{v}_L'\left(\tilde{f}_L^b\right) = 0,$$

we obtain the solution

$$\tilde{v}_L(f) = \frac{1}{\rho-\mu_L} - \frac{r+\xi}{\rho+\xi}f + \frac{r+\xi}{\rho+\xi}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}}, \qquad \tilde{f}_L^b = \frac{1}{\rho-\mu_L}\frac{\tilde{\gamma}}{\tilde{\gamma}-1}\frac{\rho+\xi}{r+\xi}$$

where

$$\tilde{\gamma} = \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho-\mu_L)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2\tilde{\gamma}^2 - \left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)\tilde{\gamma} - (\rho-\mu_L) = 0.$$

In a smooth equilibrium, $\tilde{p}_L = -\tilde{v}_L'$, and $\tilde{p}_L$ satisfies

$$(r+\xi)\,\tilde{p}_L = (r+\xi) + \left(g_L - \xi - \mu_L + \sigma^2\right) f\tilde{p}_L' + \frac{1}{2}\sigma^2 f^2 \tilde{p}_L''.$$

Differentiating once the HJB for $\tilde{v}_L$, we get

$$\tilde{g}_L = \frac{(\rho-r)\,\tilde{p}_L}{f\tilde{v}_L''}.$$

In the high state , the scaled HJB becomes

$$(\rho-\mu_H)\,\tilde{v}_H = 1 - (r+\xi)\,f - (\mu_H+\xi)\,f\tilde{v}_H' + \frac{1}{2}\sigma^2 f^2\tilde{v}_H'' + \lambda\,(\tilde{v}_L - \tilde{v}_H)\,.$$

Using conditions

$$\lim_{f \to 0} \tilde{v}_H(f) < \infty$$
$$\tilde{v}_H\left(\tilde{f}_H^b\right) = 0$$
$$\tilde{v}_H'\left(\tilde{f}_H^b\right) = 0,$$

we obtain the solution

$$\tilde{v}_H(f) = \tilde{u}_0(f) - \tilde{u}_0\left(\tilde{f}_H^b\right)\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi},$$

where

$$\tilde{u}_0(f) = \frac{1}{\rho - \mu_L}\frac{\rho + \lambda - \mu_L}{\rho + \lambda - \mu_H} - \frac{r+\xi}{\rho+\xi}f + \frac{\lambda\frac{r+\xi}{\rho+\xi}}{(\rho + \lambda - \mu_H) + \tilde{\gamma}(\mu_H + \xi) - \frac{1}{2}\sigma^2\tilde{\gamma}(\tilde{\gamma} - 1)}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}},$$

and

$$\phi = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho + \lambda - \mu_H)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2\phi^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\phi - (\rho + \lambda - \mu_H) = 0.$$

Finally, the boundary $\tilde{f}_H^b$ is pinned down by the smooth-pasting condition

$$\tilde{f}_H^b\tilde{u}_0'\left(\tilde{f}_H^b\right) - \phi\tilde{u}_0\left(\tilde{f}_H^b\right) = 0.$$

In a smooth equilibrium, $\tilde{p}_H = -\tilde{v}_H'$, and $\tilde{p}_H$ satisfies

$$(r+\xi)\tilde{p}_H = (r+\xi) + \left(g_H - \xi - \mu_H + \sigma^2\right)f\tilde{p}_H' + \frac{1}{2}\sigma^2 f^2\tilde{p}_H'' + \lambda(\tilde{p}_L - \tilde{p}_H).$$

Differentiating once the HJB for $\tilde{v}_H$, we get

$$\tilde{g}_H = \frac{(\rho - r)\tilde{p}_H}{f\tilde{v}_H''}.$$

Finally, let us compare the long-term debt prices with and without short-term debt.

$$
\begin{aligned}
J_L(X,F) = &\sup_{T,\{G_s, D_s \le J_L(X_s,F_s)\}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}\Big(X_s - (r+\xi)F_s + (\rho - y_{s-})D_{s-}\Big)ds + p_s dG_s\Big) \Big| X_t = X,\ F_t = F\right] \\
&\ge \sup_{T,\{G_s\}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}\Big(X_s - (r+\xi)F_s\Big)ds + p_s dG_s\Big) \Big| X_t = X,\ F_t = F\right] \\
&= \sup_T \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}\Big(X_s - (r+\xi)F_s\Big)ds\Big) \Big| X_t = X,\ F_t = F\right] = \tilde{V}_L(X,F) \qquad (36)
\end{aligned}
$$

where the inequality comes from that $(\rho - y_{s-})D_{s-}$ is nonnegative. It implies that the value function at $L$ stage in the benchmark is always no smaller than that in the economy with long-term debt only for any $(X,F)$. Therefore, $f_L^b \ge \tilde{f}_L^b$.

The debt price in the benchmark at $L$ stage is

$$
(r+\xi)\,p_L = (r+\xi) + \left(g_L - \xi - \mu_L + \sigma^2\right) f p_L' + \frac{1}{2}\sigma^2 f^2 p_L'',
$$

where $g_L(f) = 0$.

And in the economy with long-term debt only, the debt price is

$$
(r+\xi)\,\tilde{p}_L = (r+\xi) + \left(\tilde{g}_L - \xi - \mu_L + \sigma^2\right) f \tilde{p}_L' + \frac{1}{2}\sigma^2 f^2 \tilde{p}_L''.
$$

where $\tilde{g}_L(f) = \frac{(\rho-r)\tilde{p}_L}{f v_L''} \ge 0$.

Let $\triangle_L(f) = p_L(f) - \tilde{p}_L(f)$, which satisfies

$$
\frac{1}{2}\sigma^2 f^2 \triangle_L'' - \left(\mu_L + \xi - \sigma^2\right) f \triangle_L' - (r+\xi)\triangle_L = \tilde{g}_L f \tilde{p}_L' = -\tilde{g}_L f \tilde{v}_L'' \le 0.
$$

In addition, the debt price is

$$
\begin{aligned}
p_L(f) &= -j_L'(f) = 1 - \left(\frac{f}{f_L^b}\right)^{\gamma-1}, \\
\tilde{p}_L(f) &= -\tilde{v}_L'(f) = \frac{r+\xi}{\rho+\xi} - \frac{r+\xi}{\rho+\xi}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}-1},
\end{aligned}
$$

which implies $\triangle_L(0) = 1 - \frac{r+\xi}{\rho+\xi} > 0$ and $\triangle_L\left(\tilde{f}_L^b\right) = 1 - \left(\frac{\tilde{f}_L^b}{f_L^b}\right)^{\gamma-1} \ge 0$. Using Theorem 1, we know that $\triangle_L(f)$ can not have an interior nonpositive minimum on $\left[0, \tilde{f}_L^b\right]$. Since $\triangle_L(0) > 0$ and

$\triangle_L\left(\tilde{f}_L^b\right) \geq 0$, $\triangle_L(f) \geq 0$ for any $f \in \left[0, \tilde{f}_L^b\right]$. It implies $p_L(f) \geq \tilde{p}_L(f)$ for any $f \in \left[0, \tilde{f}_L^b\right]$. (Note we should be able to show it is strictly positive for any $f \in \left[0, \tilde{f}_L^b\right)$.)

In the high state, we know $\triangle_H(0) = p_H(f) - \tilde{p}_H(f) = 1 - \frac{r+\xi}{\rho+\xi} > 0$. Since the debt price is continuous in both economies, $\forall \varepsilon$, $\exists \delta > 0$ such that for all $0 < f < \delta$, $\triangle_H(0) > L - \varepsilon$. It implies $p_H(f) > \tilde{p}_H(f)$ when $f$ is close enough to zero. In addition, we can show that $f_H^b \geq \tilde{f}_H^b$, which implies $p\left(\tilde{f}_H^b\right) \geq 0 = \tilde{p}\left(\tilde{f}_H^b\right)$. Since the debt price is continuous in both economies, $p_H(f) \geq \tilde{p}_H(f)$ when $f$ is close enough to $\tilde{f}_H^b$. □

### A.2.1 Proof of Proposition 7

*Proof.* The proof proceeds in a few steps.

**State $\theta_t = L$.** The HJB in the low state is

$$
\begin{aligned}
(\rho + \zeta - \mu_L)\, j_L(f) &= \max_{d \leq j_L(f)} 1 - (r+\xi) f + (\rho + \zeta - y) d - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f) \\
&= 1 - (r+\xi) f + (\rho - r) j_L(f) - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f) \\
\Rightarrow (r + \zeta - \mu_L)\, j_L(f) &= 1 - (r+\xi) f - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f),
\end{aligned}
$$

where we have used the condition $y = r + \zeta$ that compensates the disaster risk. Take derivative of the above equation

$$
(r + \zeta + \xi)\, j_L'(f) = -(r+\xi) - \left(\mu_L + \xi - \sigma^2\right) f j_L''(f) + \frac{1}{2}\sigma^2 f^2 j_L'''(f).
$$

The debt price follows

$$
(r + \xi + \zeta)\, p_L(f) = (r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)
$$

From here we get,

$$
g_L(f) = \frac{(r+\xi+\zeta)\, j_L'(f) + (r+\xi) + \left(\xi + \mu_L - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f)}{f j_L''(f)} = 0.
$$

Next, we solve for $j_L(f)$, which follows

$$
(r + \zeta - \mu_L)\, j_L(f) = 1 - (r+\xi) f - (\mu_L + \xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)
$$

with boundary condition

$$j_L(f_L^b) = 0$$
$$j_L'(f_L^b) = 0.$$

The solution is

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1},$$

where

$$A_0 = \frac{1}{r+\zeta-\mu_L}$$
$$A_1 = \frac{r+\xi}{r+\zeta+\xi}$$
$$A_2 = \frac{1}{\gamma_1}\frac{r+\xi}{r+\zeta+\xi}\left(f_L^b\right)^{1-\gamma_1}$$

and

$$f_L^b = \frac{\gamma_1}{\gamma_1-1}\frac{1}{r+\zeta-\mu_L}\frac{r+\zeta+\xi}{r+\xi}$$

is the default boundary. Moreover,

$$\gamma_1 = \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\zeta-\mu_L\right)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2\gamma^2 - \left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)\gamma - (r+\zeta-\mu_L) = 0.$$

**State H.** The function of HJB and debt price are the same as benchmark so the function of long-term debt issuance is also the same, i.e, when $f < f_\dagger$,

$$g_H(f) = \frac{(\rho-r)\left(j_L'(f)-j_H'(f)\right)}{f j_H''(f)}$$

and $g_H(f) = 0$ when $f > f_\dagger$.

**Region** $(\rho+\lambda-r)\, j_L(f) \geq (\rho-r)\, j_H(f)$. The HJB is

$$(\rho+\lambda-\mu_H)\, j_H(f) = 1-(r+\xi)\, f-(\mu_H+\xi)\, f j_H'(f)+\frac{1}{2}\sigma^2 f^2 j_H''(f)+(\rho-r+\lambda) j_L(f).$$

The solution is

$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\phi_1}$$

where

$$B_0 = \frac{\rho + \lambda + \zeta - \mu_L}{(\rho + \lambda - \mu_H)(r + \zeta - \mu_L)}$$
$$B_1 = \frac{(r + \xi)(\rho + \lambda + \zeta + \xi)}{(\rho + \xi + \lambda)(r + \zeta + \xi)}$$
$$B_2 = \frac{\rho + \lambda - r}{(\rho + \lambda - \mu_H) + (\mu_H + \xi)\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1 (\gamma_1 - 1)} A_2.$$

and

$$\phi_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (\rho + \lambda - \mu_H)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2 \phi^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\phi - (\rho + \lambda - \mu_H) = 0.$$

The coefficient $B_3$ will be determined by the boundary conditions.

**Region** $(\rho + \lambda - r)\, j_L(f) < (\rho - r)\, j_H(f)$. The HJB is

$$(r + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi) f - (\mu_H + \xi) f j'_H(f) + \frac{1}{2}\sigma^2 f^2 j''_H(f).$$

The solution is

$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2},$$

where

$$D_0 = \frac{1}{r + \lambda - \mu_H}$$
$$D_1 = \frac{r + \xi}{r + \xi + \lambda}$$

and

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2 (\mu_H - r - \lambda)}}{\sigma^2} > 1,$$
$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2 (\mu_H - r - \lambda)}}{\sigma^2} < 0$$

solve

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta + \mu_H - r - \lambda = 0.$$

The coefficients $(D_2, D_3)$ will be determined by the boundary conditions.

**Boundary Conditions** The coefficients to be determined are $(B_3, D_2, D_3)$ and the free boundary $(f_\dagger, f_H^b)$; hence, we need 5 boundary conditions. The first boundary condition is the indifference condition

$$(\rho + \lambda - r)\, j_L(f_\dagger) = (\rho - r)\, j_H(f_\dagger).$$

We also have the value matching conditions

$$\begin{aligned} j_H(f_\dagger -) &= j_H(f_\dagger +) \\ j_H(f_H^b) &= 0. \end{aligned}$$

Finally, we have the smooth pasting conditions

$$\begin{aligned} j_H'(f_H^b) &= 0 \\ j_H'(f_\dagger -) &= j_H'(f_\dagger +). \end{aligned}$$

Taking together, we have the following five equations:

$$\begin{aligned} (\rho + \lambda - r)\left(A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1}\right) &= (\rho - r)\left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\phi_1}\right) \\ B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\phi_1} &= D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2} \\ D_0 - D_1 f_H^b + D_2 \left(f_H^b\right)^{\beta_1} + D_3 \left(f_H^b\right)^{\beta_2} &= 0 \\ -D_1 + \beta_1 D_2 \left(f_H^b\right)^{\beta_1 - 1} + \beta_2 D_3 \left(f_H^b\right)^{\beta_2 - 1} &= 0 \\ -B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \phi_1 B_3 (f_\dagger)^{\phi_1 - 1} &= -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1} \end{aligned}$$

The value-matching and smooth-pasting at $f_H^b$ imply that once we know $f_H^b$

$$\begin{aligned} D_2 &= \frac{\beta_2 D_0 + (1 - \beta_2) D_1 f_H^b}{\left(f_H^b\right)^{\beta_1} (\beta_1 - \beta_2)} \\ D_3 &= \frac{\beta_1 D_0 + (1 - \beta_1) D_1 f_H^b}{\left(f_H^b\right)^{\beta_2} (\beta_2 - \beta_1)}. \end{aligned}$$

From the indifference equation (the first condition), we know

$$B_3 = (f_\dagger)^{-\phi_1}\left[\frac{\rho+\lambda-r}{\rho-r}\left(A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1}\right) - \left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1}\right)\right].$$

The remaining unknowns $\left\{f_\dagger, f_H^b\right\}$ can be solved the following two equations:

$$\begin{aligned} B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\phi_1} &= D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2} \\ -B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \phi_1 B_3 (f_\dagger)^{\phi_1 - 1} &= -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1} \end{aligned}$$

**Long-term debt issuance at $f = 0$.** In the region $f < f_\dagger$, we know

$$\begin{aligned} j_H'(f) &= -B_1 + \gamma_1 B_2 f^{\gamma_1 - 1} + \phi_1 B_3 f^{\phi_1 - 1} \\ j_H''(f) &= \gamma_1 (\gamma_1 - 1) B_2 f^{\gamma_1 - 2} + \phi_1 (\phi_1 - 1) B_3 f^{\phi_1 - 2} \\ j_L'(f) &= -A_1 + \gamma_1 A_2 f^{\gamma_1 - 1}. \end{aligned}$$

Therefore,

$$\begin{aligned} g_H(f) &= \frac{(\rho - r)\left(j_L'(f) - j_H'(f)\right)}{f j_H''(f)} \\ &= \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1 (A_2 - B_2) f^{\gamma_1 - 1} - \phi_1 B_3 f^{\phi_1 - 1}\right)}{B_2 \gamma_1 (\gamma_1 - 1) f^{\gamma_1 - 1} + B_3 \phi_1 (\phi_1 - 1) f^{\phi_1 - 1}}. \end{aligned}$$

Given that $\min\{\phi_1, \gamma_1\} > 1$, $\lim_{f \to 0} g_H(f) = \infty$. Moreover,

$$g_H(f) f = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} f + \gamma_1 (A_2 - B_2) f^{\gamma_1} - \phi_1 B_3 f^{\phi_1}\right)}{B_2 \gamma_1 (\gamma_1 - 1) f^{\gamma_1 - 1} + B_3 \phi_1 (\phi_1 - 1) f^{\phi_1 - 1}}$$

Applying L'Hopital's rule to the above equation, it becomes immediately clear that as long as $\min\{\phi_1, \gamma_1\} \geq 2$, $\lim_{f \to 0} g_H(f) f > 0$. To see this, assume $\phi_1 > \gamma_1$. If $\gamma_1 > 2$,

$$\lim_{f \to 0} g_H(f) f = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1^2 (A_2 - B_2) f^{\gamma_1 - 1} - \phi_1^2 B_3 f^{\phi_1 - 1}\right)}{B_2 \gamma_1 (\gamma_1 - 1)^2 f^{\gamma_1 - 2} + B_3 \phi_1 (\phi_1 - 1)^2 f^{\phi_1 - 2}} = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda}\right)}{0} = \infty.$$

If $\gamma_1 = 2$,

$$\begin{aligned}\lim_{f\to 0} g_H(f)f &= \frac{(\rho-r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1^2\left(A_2 - B_2\right)f^{\gamma_1-1} - \phi_1^2 B_3 f^{\phi_1-1}\right)}{B_2\gamma_1(\gamma_1-1)^2 f^{\gamma_1-2} + B_3\phi_1\left(\phi_1-1\right)^2 f^{\phi_1-2}} \\ &= \frac{(\rho-r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda}\right)}{B_2\gamma_1(\gamma_1-1)^2} = \frac{(\rho-r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda}\right)}{2B_2} \\ &= \frac{2\left(\rho+\lambda-r-\zeta+\mu_H-\mu_L\right)\left(\rho-r\right)\zeta\left(r+\zeta+\xi\right)}{\left(\rho+\lambda-r\right)\left(r+\zeta-\mu_L\right)\left(r+\xi\right)\left(\rho+\xi+\lambda\right)}.\end{aligned}$$

The results are similar if $\phi_1 < \gamma_1$.

**The condition $(\rho+\lambda-r)\, j_L(0) > (\rho-r)\, j_H(0)$.** Note that the assumption $f_\dagger > 0$ requires

$$\begin{aligned} & \left(\rho+\lambda-r\right) j_L(0) > \left(\rho-r\right) j_H(0) \\ \Rightarrow\quad & \left(\rho+\lambda-r\right) A_0 - \left(\rho-r\right) B_0 > 0 \\ \Rightarrow\quad & \zeta < \frac{\lambda}{\rho-r}\left(\rho+\lambda-\mu_H\right) - \left(\mu_H-\mu_L\right).\end{aligned}$$

$\square$

## A.3 Jump Risk

The value function satisfies the equation

$$\begin{aligned}\left(\rho+\lambda-\mu\right) j(f) = 1 - \left(r+\xi\right) f - \left(\mu+\xi\right) f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) \\ + \max\left\{(\rho-r)j(\eta f) + \lambda j(\eta f), (\rho-r)j(f)\right\}\end{aligned}$$

On the region $(f_\dagger, f^b)$, the equation reduces to

$$\left(r+\lambda-\mu\right) j(f) = 1 - \left(r+\xi\right) f - \left(\mu+\xi\right) f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f),$$

while on the region $(f_\dagger, f^b)$, the equation reduces to

$$\left(\rho+\lambda-\mu\right) j(f) = 1 - \left(r+\xi\right) f - \left(\mu+\xi\right) f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) + (\rho+\lambda-r)j(\eta f).$$

The threshold $f_\dagger$ is determined by the condition

$$(\rho + \lambda - r) j(\eta f_\dagger) = (\rho - r) j(f_\dagger)$$

**Solution Value Function**

The first step is to solve the value function on the interval $(f_\dagger, f^b)$. This a a standard second order differential equation whose solution is given by

$$j(f) = \frac{1}{r + \lambda - \mu} - \frac{r + \xi}{r + \lambda + \xi} f + C_1 f^{\psi_1} + C_2 f^{\psi_2},$$

where

$$(r + \lambda - \mu) = -\left(\mu + \xi + \frac{1}{2}\sigma^2\right)\psi + \frac{1}{2}\sigma^2\psi^2$$

$$\psi_1 = \frac{\left(\mu + \xi + \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 \left(r + \lambda - \mu\right)}}{\sigma^2}$$

$$\psi_2 = \frac{\left(\mu + \xi + \frac{1}{2}\sigma^2\right) - \sqrt{\left(\mu + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 \left(r + \lambda - \mu\right)}}{\sigma^2}$$

Given a default threshold $f^b$ satisfying the value matching and smooth pasting condition, we can solve for the constants $C_1$ and $C_2$ as a function of $f^b$, by solving the system

$$\frac{1}{r + \lambda - \mu} - \frac{r + \xi}{r + \lambda + \xi} f^b + C_1 \cdot (f^b)^{\psi_1} + C_2 \cdot (f^b)^{\psi_2} = 0$$

$$-\frac{r + \xi}{r + \lambda + \xi} + \psi_1 C_1 \cdot (f^b)^{\psi_1 - 1} + \psi_2 \cdot C_2 (f^b)^{\psi_2 - 1} = 0$$

Let $C_1(f^b)$ and $C_2(f^b)$ be the solution to the previous equation as a function of $f^b$, so the value function is

$$j(f) = \frac{1}{r + \lambda - \mu} - \frac{r + \xi}{r + \lambda + \xi} f + C_1(f^b) f^{\psi_1} + C_2(f^b) f^{\psi_2}.$$

From condition determining the threshold $f_\dagger$, it follows immediately that $f_\dagger < \eta^{-1} f^b$. By treating $j(\eta f)$ as an exogenously given function, we can treat the equation on $(0, f_\dagger)$ as an ordinary second order differential equation. The solution to such equation takes the form

$$j(f) = \frac{1}{\rho + \lambda - \mu} - \frac{r + \xi}{\rho + \lambda + \xi} f + \vartheta(f) + A_1 \varphi_1(f) + A_2 \varphi_2(f),$$

where $\varphi_1(f)$ and $\varphi_2(f)$ are solutions to the homogenous equation

$$(\rho+\lambda-\mu)\,\varphi(f) = -\,(\mu+\xi)\,f\varphi'(f) + \frac{1}{2}\sigma^2 f^2 \varphi''(f).$$

It follows from here that $\varphi(f) = f^{\upsilon}$ so

$$j(f) = \frac{1}{\rho+\lambda-\mu} - \frac{r+\xi}{\rho+\lambda+\xi} f + \vartheta(f) + A_1 f^{\upsilon_1} + A_2 f^{\upsilon_2},$$

where

$$\upsilon_1 = \frac{\left(\mu+\xi+\frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu+\xi+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho+\lambda-\mu\right)}}{\sigma^2}$$
$$\upsilon_2 = \frac{\left(\mu+\xi+\frac{1}{2}\sigma^2\right) - \sqrt{\left(\mu+\xi+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho+\lambda-\mu\right)}}{\sigma^2},$$

and the function $\vartheta(f)$ can be obtained using the Wronskian of the homogenous equation, which is given by

$$W(f) = \varphi_1(f)\varphi_2'(f) - \varphi_1'(f)\varphi_2(f) = \upsilon_2 f^{\upsilon_1} f^{\upsilon_2-1} - \upsilon_1 f^{\upsilon_1-1} f^{\upsilon_2} = (\upsilon_2-\upsilon_1) f^{\upsilon_1+\upsilon_2-1},$$

as follows:

$$\begin{aligned}
\vartheta(f) &= -f^{\upsilon_1}\int_{\eta^{-n}f_\dagger}^{f} \frac{y^{\upsilon_2(\rho+\lambda-r)j(\eta y)}}{\frac{1}{2}\sigma^2 y^2 W(y)}dy - f^{\upsilon_2}\int_{f}^{\eta^{-(n-1)}f_\dagger} \frac{y^{\upsilon_1}(\rho+\lambda-r)j(\eta y)}{\frac{1}{2}\sigma^2 y^2 W(y)}dy \\
&= f^{\upsilon_1}\int_{\eta^{-n}f_\dagger}^{f} \frac{(\rho+\lambda-r)j(\eta y)}{\frac{1}{2}\sigma^2(\upsilon_1-\upsilon_2)y^{\upsilon_1+1}}dy + f^{\upsilon_2}\int_{f}^{\eta^{-(n-1)}f_\dagger} \frac{(\rho+\lambda-r)j(\eta y)}{\frac{1}{2}\sigma^2(\upsilon_1-\upsilon_2)y^{\upsilon_2+1}}dy
\end{aligned}$$

Thus, letting

$$j_0(f) \equiv \frac{1}{r+\lambda-\mu} - \frac{r+\xi}{r+\lambda+\xi} f + C_1(f^b) f^{\psi_1} + C_2(f^b) f^{\psi_2},$$

we can compute the solution on $f \in (\eta^{-n}f_\dagger, \eta^{-(n-1)}f_\dagger)$ recursively as

$$\begin{aligned}
j_n(f) = \frac{1}{\rho+\lambda-\mu} - \frac{r+\xi}{\rho+\lambda+\xi} f + \frac{2(\rho+\lambda-r)}{\sigma^2(\upsilon_1-\upsilon_2)}\left[\int_{\eta^{-n}f_\dagger}^{f}\left(\frac{f}{y}\right)^{\upsilon_1}\frac{j_{n-1}(\eta y)}{y}dy + \int_{f}^{\eta^{-(n-1)}f_\dagger}\left(\frac{f}{y}\right)^{\upsilon_2}\frac{j_{n-1}(\eta y)}{y}dy\right] \\
+ A_1^n f^{\upsilon_1} + A_2^n f^{\upsilon_2}
\end{aligned}$$

where the constants $A_1^n$ and $A_2^n$ are such

$$j_n(\eta^{-(n-1)} f_\dagger) = j_{n-1}(\eta^{-(n-1)} f_\dagger)$$
$$j_n'(\eta^{-(n-1)} f_\dagger) = j_{n-1}'(\eta^{-(n-1)} f_\dagger).$$

From here, we get that the condition for the threshold $f_\dagger$ can be written as

$$\frac{(\rho+\lambda-r)}{\rho-r} j_0(\eta f_\dagger) = \frac{1}{\rho+\lambda-\mu} - \frac{r+\xi}{\rho+\lambda+\xi} f_\dagger + \frac{2(\rho+\lambda-r)}{\sigma^2(\upsilon_1-\upsilon_2)} \int_{\eta^{-1} f_\dagger}^{f_\dagger} \left(\frac{f}{y}\right)^{\upsilon_1} \frac{j_0(\eta y)}{y} dy$$
$$+ A_1^1 f_\dagger^{\upsilon_1} + A_2^1 f_\dagger^{\upsilon_2}$$

Finally, the transversality condition requires that $\lim_{n\to\infty} A_2^n = 0$.